INSTITUTE OF
MATHEMATICS &
ITS APPLICATIONS

Vectors – Tutor Notes

This session builds on the vector concepts introduced in the prereading, providing additional examples and explanations to reinforce understanding. Begin by assessing students' current knowledge – ask what they recall about vectors from A-level or from the prereading. Use this to identify which sections may need clarification. If students express uncertainty about a particular topic, or indicate they found a section challenging, you can navigate directly to the relevant guidance below. The material is structured to follow the prereading sequence, allowing you to address specific gaps based on your student cohort's confidence levels.

Part I

Welcome Back – 15 Minutes

Scalars and vectors: Start by asking students to distinguish between scalars and vectors. Use everyday examples: temperature (scalar), wind speed and direction (vector). Emphasise that vectors require both magnitude and direction for complete description. Within this discussion, establish the difference between mass and weight – mass (kg) is a scalar quantity, while weight (N) is a force vector acting downward due to gravity. This naturally leads to the understanding that all forces, including weight, are vectors!

Representing vectors geometrically: Draw an arrow and remind students that the length represents magnitude, the arrowhead shows direction. Ensure students are confident with this visual representation – it underpins everything that follows, from addition to resolution and beyond.

Unit vectors and basis vectors: Introduce $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ as the building blocks. Show how any 2D vector can be expressed as a combination of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. The hat notation indicates a vector of length 1.

Component form: Demonstrate writing vectors as $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}$. Work through simple examples: $\mathbf{v} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$. Show column vector notation as an alternative. It may be worthwhile showing that 3D vectors follow a similar representation, simply adding a $v_z\hat{\mathbf{k}}$ component (or a third entry in column form).

Magnitude of a vector: Remind students of Pythagoras: $|\mathbf{v}| = \sqrt{v_x^2 + v_y^2}$.

Vector addition and subtraction: Demonstrate tip-to-tail addition geometrically, then show component-wise addition algebraically. Though not explicitly covered in the prereading, tutors may wish to contrast this with the parallelogram method (completing the parallelogram to find the diagonal resultant). Emphasise that regardless of the geometric visualisation, vectors add component by component algebraically.

Scalar multiplication: Show that multiplying by a scalar changes the length, and a negative scalar reverses direction. Use $2\mathbf{v}$ and $-\mathbf{v}$ as examples. Use this opportunity to emphasise verbally the idea of parallel vectors – any scalar multiple of $\mathbf{v}$ is parallel to $\mathbf{v}$ (or anti-parallel if the scalar is negative).

Resolving vectors: This is perhaps the most practically important section. Draw a vector at an angle θ to the horizontal, showing $v_x = |\mathbf{v}|\cos\theta$ and $v_y = |\mathbf{v}|\sin\theta$. Emphasise that the choice of sine or cosine depends on which angle is given. Briefly introduce static equilibrium problems where forces are resolved and summed to zero.

Position vectors and displacement: Clarify the difference between a position vector (location relative to origin) and a displacement vector (change in position). Show $\overrightarrow{AB} = \mathbf{r}_B - \mathbf{r}_A$.

Direction and unit vectors: Show how to find a unit vector in any direction: $\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$. Take this apart carefully – the numerator is the vector itself, and the denominator is its magnitude. Dividing a vector by its size yields a vector of length 1 in the same direction; for example, a vector of length 4, when divided by 4, gives a vector pointing the same way but with magnitude 1.

Dot product: This is likely one of the newer concepts for students, so take time to develop it carefully. Introduce both the geometric form $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$ and the component form $\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$. Emphasise that these are equivalent – the geometric form reveals the meaning (product of magnitudes times cosine of the angle), while the component form is what we actually compute with coordinates. Highlight the two key applications: finding angles between vectors ($\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$), and testing perpendicularity (if $\mathbf{a} \cdot \mathbf{b} = 0$, the vectors are perpendicular). The geometric interpretation – the dot product measures how much one vector extends in the direction of another, i.e., the projection of $\mathbf{b}$ onto $\mathbf{a}$ has length $|\mathbf{b}| \cos\theta$ – is worth illustrating with a diagram to make the concept concrete.

Part II

Getting Started – 10 Minutes

Let students work on **Question 1**, then walk through the answers. Students may need reminding that $\mathbf{E}_1$ is purely horizontal, so its components are $(200, 0)$. For $\mathbf{E}_2$, ensure they correctly apply $E_{2x} = 150 \cos 60°$ and $E_{2y} = 150 \sin 60°$. Watch for the common mistake of swapping sine and cosine. The resultant magnitude calculation uses Pythagoras, and the direction uses $\tan\theta = E_y/E_x$. Tutors may choose to draw the vector triangle more clearly, labelling the horizontal and vertical components to help students visualise the resultant. Emphasise that the final answer should include both magnitude and direction – a vector quantity requires both.

Getting Stuck In – 30 Minutes

Ask students to focus on **Questions 2 to 4**. Pay special attention to:

- **Q2:** Students may need help constructing the free-body diagram and correctly resolving the tension forces. The angles are given relative to vertical, so emphasise that the horizontal components are $T \sin\theta$ and vertical components are $T \cos\theta$. Watch for sign conventions in the horizontal equation: leftward forces are negative if right is positive. In part (d), emphasise the reasoning that as F increases, T_B increases while T_A decreases – this determines which tether reaches its limit first. Part (e) requires qualitative reasoning; encourage students to think about how changing each angle affects the horizontal pull of each tether. Increasing the right tether angle makes it more effective at opposing the gust, so $F_{\max}$ increases.
- **Q3:** Students may find 3D vectors intimidating – reassure them that it's just an extension of 2D with an extra component. For the angle between panels, emphasise that the dot product $\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$ is the key calculation. Part (c) tests understanding that "power is proportional to the cosine of the angle between the panel's normal and the sunlight direction" – this is effectively the dot product with the sunlight direction $\mathbf{s} = \mathbf{k}$. Ensure students unpack this idea: for panel A, $\mathbf{a} \cdot \mathbf{s} = 1/\sqrt{3} \approx 0.577$, so it collects about 57.7% of maximum possible power. Panel B collects zero because its normal has no $\mathbf{k}$ component – some students may recognise this intuitively without calculation. Part (d) is a straightforward recalculation with the new vector.
- **Q4:** Spend time ensuring students understand why the components become $v_x = 300 \sin\theta$ (east) and $v_y = 300 \cos\theta$ (north) – draw a diagram showing the heading angle measured from north, and resolve accordingly. The condition for ground track due north ($300 \sin\theta + 50 = 0$) is a key insight. Emphasise that the wind adds to the east component, so to cancel it the aircraft must point west of north, making $\sin\theta$ negative. Part (d) extends the idea to the return journey – note the symmetry: the required heading is $180° + 9.6° = 189.6°$. The ground speed and time are the same as outbound. The total time increase of about 3.4 minutes is small but noticeable and this offers a good discussion

point about real-world navigation: is this level of time difference significant? How accurate is the model?

Break

Encourage students to step away from screens briefly.

Part III

More Problem Solving – 30 Minutes

Ask students to focus on **Questions 5 to 7** in that order.

- **Q5:** Students may find the momentum-change formula $\mathbf{F} = \dot{m}_C\mathbf{v}_C - \dot{m}_A\mathbf{v}_A - \dot{m}_B\mathbf{v}_B$ unfamiliar. Emphasise that it's simply applying Newton's second law to fluid flow – the force equals the rate of change of momentum. The algebra with π and fractions can get messy – encourage neat, step-by-step working, keeping π symbolic until the final numerical step. Pay special attention to the vector directions: $\mathbf{v}_A$ is along $\mathbf{i}$, $\mathbf{v}_B$ has both components, and $\mathbf{v}_C$ is along $\mathbf{j}$. Ensure students correctly compute $\dot{m}_B\mathbf{v}_B$. The final angle $\theta \approx 134.4^\circ$ (measured from positive x-axis) may surprise students – remind them that negative F_x and positive F_y place the vector in the second quadrant. Part (d) reinforces Newton's third law – the external force needed to hold the junction is equal and opposite to $\mathbf{F}$. Draw the force diagram clearly.
- **Q6:** Parts (a)-(c) are straightforward applications of $\mathbf{s} = \mathbf{v}t + \frac{1}{2}\mathbf{a}t^2$. Emphasise working in components – treat x and y separately. Part (d) requires finding a unit vector and using the dot product to find the angle. Encouraging students to sketch the vectors out helps visualise why the angle comes out as 55.3° and why this exceeds the 30° tolerance. Part (e) introduces an important subtlety: checking only the start and end distances is not sufficient to guarantee that no collision occurred during motion. This is a valuable insight about collision detection in game programming, robotics, and any application involving moving objects. Ensure students understand this nuance – it adds a genuinely interesting dimension to what might otherwise seem like a routine distance calculation.
- **Q7:** Students may need help visualising how $\mathbf{a}$ and $\mathbf{b}$ are defined. Draw the hexagonal lattice and show the basis vectors at 120°. Part (b) requires combining vectors algebraically. Emphasise collecting $\mathbf{i}$ and $\mathbf{j}$ terms separately. Part (c) derives a general distance formula $d = a\sqrt{m^2 + n^2 - mn}$. This is a powerful result – students should appreciate how the dot product simplifies the algebra.

Wrap Up – 5 Minutes

Pose any remaining questions as extensions:

- **Q8:** This question introduces deviation vectors and tolerances. Part (a) is straightforward vector subtraction. Parts (b)-(d) check various tolerances: position tolerance, distance tolerance, and angular tolerance. Emphasise that each must be satisfied independently for the part to be acceptable. Part (e) on the mean deviation vector is particularly critical as a non-zero average indicates systematic machine error. Ensure students can interpret this result – the $0.2\mathbf{j}$ mm offset is purely vertical, meaning the machine is consistently drilling 0.2 mm too far in the y-direction. This insight is helpful for calibrating the machine to correct the error.
- **Q9:** Parts (a)-(d) are straightforward magnitude and angle calculations. Part (e) asks for a qualitative explanation. Look for answers that connect steeper angles to faster descent through denser atmosphere. Part (f) is an optimisation-style question comparing two adjustment strategies to achieve $\phi = 3.0^\circ$. Guide students to set up the condition $\tan 3.0^\circ = v'_y/v'_x$ and explore both options. The vector diagram (or calculation) clearly shows that decreasing the vertical component requires a much smaller change than increasing the horizontal component. This is a valuable insight about efficient manoeuvring – small adjustments to the vertical component are far more efficient for modifying flight-path angle than large changes to horizontal speed. Discuss why this matters for spacecraft control: fuel is limited, so efficient adjustments are critical.

Remind students that they can attempt these in their own time. Encourage them to use the solutions only after attempting the problems themselves.