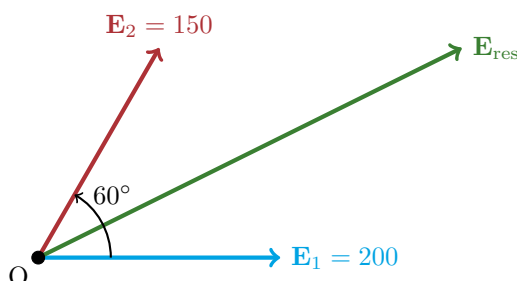




Vectors – Solutions

1. A uniform electric field \mathbf{E}_1 acts horizontally to the right with magnitude 200 V m^{-1} , and at the same point a second field \mathbf{E}_2 has magnitude 150 V m^{-1} at 60° above the horizontal. By expressing \mathbf{E}_1 and \mathbf{E}_2 as 2D vectors in component form, find the magnitude and direction of the resultant electric field at this point.

Solution:



Taking the positive x -axis as horizontal to the right and the positive y -axis as vertically upward,

$$\mathbf{E}_1 = (200, 0) \text{ V m}^{-1}.$$

For \mathbf{E}_2 , the components are

$$E_{2x} = 150 \cos 60^\circ = 75 \text{ V m}^{-1}, \quad E_{2y} = 150 \sin 60^\circ = 75\sqrt{3} \text{ V m}^{-1}.$$

Thus

$$\mathbf{E}_2 = (75, 75\sqrt{3}) \text{ V m}^{-1}.$$

The resultant $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ has components

$$E_x = 200 + 75 = 275 \text{ V m}^{-1}, \quad E_y = 0 + 75\sqrt{3} = 75\sqrt{3} \text{ V m}^{-1}.$$

The magnitude of the resultant is then calculated as,

$$E = \sqrt{E_x^2 + E_y^2} = \sqrt{275^2 + (75\sqrt{3})^2} = 304.14 \text{ V m}^{-1}.$$

The angle θ above the horizontal is given by

$$\tan \theta = \frac{E_y}{E_x} = \frac{75\sqrt{3}}{275}.$$

$$\theta \approx 25.3^\circ.$$

$$\boxed{E = 304.14 \text{ V m}^{-1}, \quad \theta \approx 25.3^\circ \text{ above horizontal}}$$

2. During a storm, a 100 N medical-delivery drone is attempting to hover above a certain location. To prevent it from being blown to the left by high winds, two emergency tethers are quickly attached to nearby anchor points above, and on opposite sides of, the drone.

→ Tether A, anchored to the left, makes 30° right of vertical.

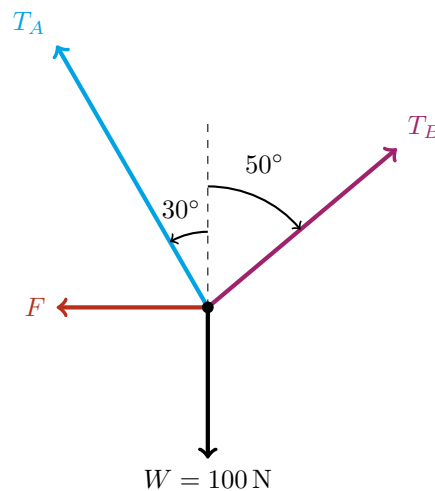
- Tether B, anchored to the right, makes 50° left of vertical.
 → A violent gust exerts a steady horizontal force F to the left on the drone.

The system is in static equilibrium.

- Construct a free-body diagram for the drone and write the vector equation for static equilibrium.
- By resolving forces into horizontal and vertical components, derive the two scalar equilibrium equations.
- For a measured gust force of 40 N, determine the tension in each tether, T_A and T_B .
- Given that both tethers have a maximum safe working load of 120 N, calculate the maximum gust force F_{\max} that the system can withstand. State which tether would fail first if this limit were exceeded.
- Would increasing the 30° left tether angle or the 50° right tether angle allow the system to withstand a larger maximum gust force F_{\max} ? Provide a brief justification.

Solution:

(a) Free-body diagram and vector equilibrium equation



The sum of all forces must be zero:

$$\vec{T}_A + \vec{T}_B + \vec{F} + \vec{W} = \vec{0}.$$

(b) Scalar equilibrium equations

Resolve forces into horizontal (x positive to the right) and vertical (y positive upward) components.

$$\begin{aligned} T_A &: (-T_A \sin 30^\circ, T_A \cos 30^\circ), \\ T_B &: (T_B \sin 50^\circ, T_B \cos 50^\circ), \\ F &: (-F, 0), \\ W &: (0, -100). \end{aligned}$$

Setting the sum to zero gives:

$$\begin{cases} -T_A \sin 30^\circ + T_B \sin 50^\circ - F = 0, \\ T_A \cos 30^\circ + T_B \cos 50^\circ - 100 = 0. \end{cases}$$

(c) Tensions for $F = 40$ N

Simplifying the two scalar equilibrium equations using the trigonometric ratios, $\sin 30^\circ = 0.5$, $\cos 30^\circ \approx 0.8660$, $\sin 50^\circ \approx 0.7660$, $\cos 50^\circ \approx 0.6428$:

$$\begin{cases} -0.5T_A + 0.7660T_B = F, \\ 0.8660T_A + 0.6428T_B = 100. \end{cases}$$

Substitute $F = 40$ into the first equation:

$$0.7660T_B - 0.5T_A = 40 \quad \Rightarrow \quad T_A = 1.532T_B - 80.$$

Substitute this into the second equation:

$$0.8660(1.532T_B - 80) + 0.6428T_B = 100,$$

$$1.327T_B - 69.28 + 0.6428T_B = 100,$$

$$1.9698T_B = 169.28 \quad \Rightarrow \quad T_B \approx 86.0 \text{ N}.$$

Then $T_A = 1.532 \times 86.0 - 80 \approx 51.8 \text{ N}$.

$$\boxed{T_A \approx 51.8 \text{ N}, \quad T_B \approx 86.0 \text{ N}}.$$

(d) Maximum gust force with safe working load 120 N

Tether A pulls up and to the left, while tether B pulls up and to the right (opposing the gust). As the leftward gust force F increases, tether B must work harder to counteract it, so its tension T_B increases. Conversely, tether A is actually pulling in the same direction as the gust (leftwards), so its tension T_A decreases as the gust grows. Therefore, tether B will be the first to reach the safe working limit of 120 N. We set $T_B = 120$ and solve for the corresponding F .

From the equilibrium equations derived in part (b):

$$\begin{cases} -0.5T_A + 0.7660T_B = F, \\ 0.8660T_A + 0.6428T_B = 100. \end{cases}$$

Substitute $T_B = 120$ into the first equation:

$$-0.5T_A + 0.7660(120) = F \quad \Rightarrow \quad -0.5T_A + 91.92 = F.$$

Substitute $T_B = 120$ into the second equation:

$$0.8660T_A + 0.6428(120) = 100 \quad \Rightarrow \quad 0.8660T_A + 77.136 = 100.$$

Solve the second equation for T_A :

$$0.8660T_A = 22.864 \quad \Rightarrow \quad T_A = \frac{22.864}{0.8660} \approx 26.4 \text{ N}.$$

Now use the first equation to find F :

$$F = -0.5(26.4) + 91.92 = -13.2 + 91.92 = 78.72 \text{ N}.$$

Thus the maximum safe gust force is

$$\boxed{F_{\max} \approx 78.7 \text{ N}},$$

(e) Effect of increasing the angles

Tether A pulls up and to the left, actually assisting the gust, while tether B pulls up and to the right, opposing the gust.

- Increasing the left tether angle (30°) makes it pull more horizontally leftward, increasing its contribution to the leftward force. This means tether B must work even harder to counteract both the gust and tether A's increased leftward pull, so the maximum gust force decreases.
- Increasing the right tether angle (50°) makes it pull more horizontally rightward, making it more effective at opposing the gust. This allows a larger gust force before tether B reaches its 120 N limit, so F_{\max} increases.

Therefore, **increasing the 50° right tether angle** allows the system to withstand a larger maximum gust force, while increasing the left tether angle would reduce it.

3. A communication satellite has two solar panels. Their orientation vectors (pointing normal to the panel surfaces) are given by:

$$\mathbf{a} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}), \quad \mathbf{b} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$$

- (a) Show that both \mathbf{a} and \mathbf{b} are unit vectors.
- (b) Calculate the angle between the two panels.
- (c) Sunlight arrives along the direction $\mathbf{s} = \mathbf{k}$. The solar power collected by a panel is proportional to the cosine of the angle between the panel's normal and the sunlight direction. Use dot products to determine the power collected by each panel.
- (d) During a manoeuvre, Panel A is rotated so that its new orientation is $\mathbf{a}' = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k})$. Find the new angle between Panel A' and Panel B.

Solution:

(a) Show that \mathbf{a} and \mathbf{b} are unit vectors

Compute the magnitudes:

$$|\mathbf{a}| = \frac{1}{\sqrt{3}}\sqrt{1^2 + 1^2 + 1^2} = \frac{1}{\sqrt{3}}\sqrt{3} = 1,$$

$$|\mathbf{b}| = \frac{1}{\sqrt{2}}\sqrt{1^2 + 1^2} = \frac{1}{\sqrt{2}}\sqrt{2} = 1.$$

Thus both are unit vectors.

(b) Angle between the two panels

The cosine of the angle θ between the normals is computed through dot products:

$$\cos \theta = \mathbf{a} \cdot \mathbf{b} = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}}(1 \cdot 1 + 1 \cdot 1 + 1 \cdot 0) = \frac{1}{\sqrt{6}}(1 + 1 + 0) = \sqrt{\frac{2}{3}}.$$

Hence

$$\theta = \arccos \sqrt{\frac{2}{3}} \approx 35.3^\circ.$$

$$\boxed{\theta \approx 35.3^\circ}.$$

(c) Power collected from sunlight along $\mathbf{s} = \mathbf{k}$

The power is proportional to the cosine of the angle between the panel normal and the sunlight direction, i.e. the dot product with \mathbf{s} (since both are unit vectors). For panel A:

$$\mathbf{a} \cdot \mathbf{s} = \frac{1}{\sqrt{3}}(1 \cdot 0 + 1 \cdot 0 + 1 \cdot 1) = \frac{1}{\sqrt{3}} \approx 0.577.$$

For panel B:

$$\mathbf{b} \cdot \mathbf{s} = \frac{1}{\sqrt{2}}(1 \cdot 0 + 1 \cdot 0) = 0.$$

The factor is infinite (since Panel B collects zero because its normal is perpendicular to the sunlight direction), but more meaningfully, Panel A collects 57.7% of the maximum possible (if it were directly facing the sun).

(d) New orientation after manoeuvre

Panel A is rotated to $\mathbf{a}' = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k})$. The new angle ϕ with panel B is given by

$$\cos \phi = \mathbf{a}' \cdot \mathbf{b} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(1 \cdot 1 + 0 \cdot 1 + 1 \cdot 0) = \frac{1}{2}(1 + 0 + 0) = \frac{1}{2}.$$

Hence $\phi = \arccos \frac{1}{2} = 60^\circ$.

$$\boxed{\phi = 60^\circ}.$$

4. An aircraft has an airspeed (speed relative to the air) of 300 km h^{-1} . It needs to fly from Cardiff to Edinburgh, which is 600 km due north. Use a Cartesian coordinate system where the unit vector \mathbf{i} points east and \mathbf{j} points north.
- If there is no wind, calculate the time taken for the flight.
 - A wind of speed 50 km h^{-1} is blowing from the west.
 - The pilot points the aircraft due north. Write down the air velocity vector and the wind velocity vector. Hence find the ground velocity vector and the ground speed.
 - After 1 hour of flying, what is the aircraft's position relative to Cardiff? How far east has the aircraft been blown off course?
 - To reach Edinburgh, the pilot must adjust the heading so that the ground track is directly north. Let the heading be an angle θ measured clockwise from north.
 - Find the value of θ needed to cancel the effect of the wind and give a ground velocity that is due north.
 - Calculate the resulting ground speed and the time for the flight.
 - For the return flight from Edinburgh to Cardiff, the wind is still 50 km h^{-1} from the west. The pilot wants to fly directly south (ground track due south).
 - Determine the required heading φ (measured clockwise from north) to achieve this.
 - Compare the total time for the round trip (outbound and return) with the total time if there were no wind. Comment briefly.

Solution:

(a) Time of flight with no wind

With no wind, ground speed equals airspeed. Time is simply

$$t = \frac{\text{distance}}{\text{speed}} = \frac{600}{300} = 2 \text{ hours.}$$

$$\boxed{t = 2 \text{ h}}$$

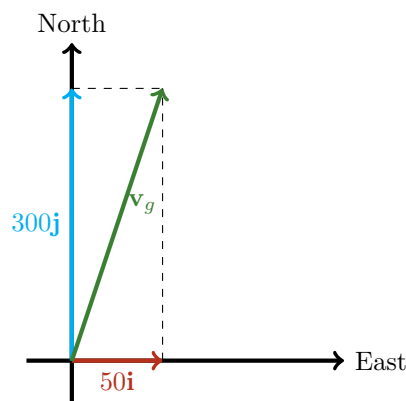
(b) Wind of 50 km h^{-1} from the west

Wind from the west means it blows toward the east, so wind velocity vector is $\mathbf{w} = 50\mathbf{i}$.

(i) Pilot points aircraft due north

Air velocity (relative to air) is $\mathbf{v}_a = 300\mathbf{j}$. Ground velocity is the vector sum:

$$\mathbf{v}_g = \mathbf{v}_a + \mathbf{w} = 300\mathbf{j} + 50\mathbf{i}.$$



The ground speed is the magnitude:

$$|\mathbf{v}_g| = \sqrt{50^2 + 300^2} = 50\sqrt{37} \approx 304.1 \text{ km h}^{-1}.$$

(ii) Position after 1 hour

Position relative to Cardiff: $\mathbf{r} = \mathbf{v}_g \times 1 = (50, 300) \text{ km}$. Thus the aircraft is 300 km north

and 50 km east of Cardiff. The eastward drift off the direct north line is 50 km.

$$\boxed{\text{Position: } (50, 300) \text{ km, } \text{drift} = 50 \text{ km}}.$$

(c) Heading for a ground track due north

To counteract the wind, the pilot must point the aircraft slightly west of north so that the wind drift cancels the eastward component. Let the heading be angle θ measured clockwise from north. Then the air velocity is

$$\mathbf{v}_a = 300(\sin \theta \mathbf{i} + \cos \theta \mathbf{j}).$$

Adding wind $\mathbf{w} = 50\mathbf{i}$ gives ground velocity

$$\mathbf{v}_g = (300 \sin \theta + 50) \mathbf{i} + 300 \cos \theta \mathbf{j}.$$

We require the ground track to be due north, i.e. the east component zero:

$$300 \sin \theta + 50 = 0 \implies \sin \theta = -\frac{50}{300} = -\frac{1}{6}.$$

Since θ is measured clockwise from north, a negative sine means the heading is west of north. The principal value is $\theta = \arcsin(-1/6) \approx -9.59^\circ$, but in navigation we use $360^\circ - 9.59^\circ = 350.41^\circ$ (or 9.6° west of north).

The north component gives the ground speed:

$$v_g = 300 \cos \theta = 300\sqrt{1 - \sin^2 \theta} = 50\sqrt{35} \approx 295.8 \text{ km h}^{-1}.$$

The time for the flight is

$$t = \frac{600}{v_g} = \frac{600}{50\sqrt{35}} \approx 2.028 \text{ hours.}$$

$$\boxed{\theta = 350.4^\circ \text{ (or } 9.6^\circ \text{ W of N), } v_g = 50\sqrt{35} \text{ km h}^{-1}, t = \frac{12}{\sqrt{35}} \text{ h.}}$$

(d) Return flight southwards

Wind still from west, $\mathbf{w} = 50\mathbf{i}$. Let the heading be φ clockwise from north. Air velocity: $\mathbf{v}_a = 300(\sin \varphi \mathbf{i} + \cos \varphi \mathbf{j})$. Ground velocity:

$$\mathbf{v}_g = (300 \sin \varphi + 50) \mathbf{i} + 300 \cos \varphi \mathbf{j}.$$

We want the ground track due south, i.e. east component zero and north component negative:

$$300 \sin \varphi + 50 = 0 \implies \sin \varphi = -\frac{1}{6},$$

and $300 \cos \varphi < 0$ (southward). The angle satisfying $\sin \varphi = -1/6$ with $\cos \varphi < 0$ is

$$\varphi = 180^\circ + \arcsin \frac{1}{6} \approx 180^\circ + 9.59^\circ = 189.59^\circ,$$

i.e. 9.6° west of south (measured clockwise from north, this is 189.6°).

The ground speed south is the magnitude of the north component:

$$v_g = |300 \cos \varphi| = 300 \cos 9.59^\circ = 50\sqrt{35} \text{ km h}^{-1} \text{ (same as outbound).}$$

$$\text{Time for return: } t = \frac{600}{50\sqrt{35}} = \frac{12}{\sqrt{35}} \text{ h.}$$

Total round-trip time with wind:

$$T_{\text{wind}} = 2 \times \frac{12}{\sqrt{35}} = \frac{24}{\sqrt{35}} \text{ h} \approx 4.057 \text{ h.}$$

Without wind, total time would be $2 \times 2 = 4$ h. The wind increases the journey by about 0.057 h.

$$\varphi = 189.6^\circ, \quad T_{\text{wind}} = \frac{24}{\sqrt{35}} \text{ h}.$$

5. Water flows through a horizontal pipe system with two branches meeting at a junction J . The flow is steady (not changing with time) and incompressible (water density is constant, $\rho = 1000 \text{ kg m}^{-3}$). At junction J :

- Pipe A (inlet): diameter 0.1 m, water velocity 2 m s^{-1} directed along the unit vector $\boldsymbol{\mu}_A = \mathbf{i}$
- Pipe B (inlet): diameter 0.08 m, water velocity 1.5 m s^{-1} directed along $\boldsymbol{\mu}_B = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$
- Pipe C (outlet): diameter 0.12 m, water exits along $\boldsymbol{\mu}_C = \mathbf{j}$

- (a) The volume flow rate Q (in $\text{m}^3 \text{ s}^{-1}$) in a pipe is given by $Q = Av$, where A is the cross-sectional area. Calculate Q_A and Q_B . For incompressible steady flow, the total inflow equals the total outflow. Use this to find the speed v_C in Pipe C.
- (b) Write the velocity vectors \mathbf{v}_A , \mathbf{v}_B , \mathbf{v}_C in component form.
- (c) The mass flow rate is $\dot{m} = \rho Q$. The force exerted by the water on the junction is given by the momentum-change formula:

$$\mathbf{F} = \dot{m}_C \mathbf{v}_C - \dot{m}_A \mathbf{v}_A - \dot{m}_B \mathbf{v}_B$$

Calculate the force vector \mathbf{F} (in N). Find its magnitude and the angle it makes with the x -axis.

- (d) If the junction is anchored so that it does not move, what external force must be applied to the junction? Sketch the forces on the junction, indicating the direction of \mathbf{F} .

Solution:

(a) Volume flow rates and outlet speed

Pipe cross-sectional areas:

$$A_A = \pi \left(\frac{0.1}{2}\right)^2 = \frac{\pi}{400} \text{ m}^2, \quad A_B = \pi \left(\frac{0.08}{2}\right)^2 = \frac{\pi}{625} \text{ m}^2, \quad A_C = \pi \left(\frac{0.12}{2}\right)^2 = \frac{9\pi}{2500} \text{ m}^2.$$

Volume flow rates (inflow):

$$Q_A = A_A v_A = \frac{\pi}{400} \times 2 = \frac{\pi}{200} \text{ m}^3 \text{ s}^{-1}, \quad Q_B = A_B v_B = \frac{\pi}{625} \times 1.5 = \frac{3\pi}{1250} \text{ m}^3 \text{ s}^{-1}.$$

$$\text{Total inflow} = Q_A + Q_B = \frac{\pi}{200} + \frac{3\pi}{1250} = \frac{37\pi}{5000} \text{ m}^3 \text{ s}^{-1}.$$

For steady incompressible flow, inflow equals outflow: $Q_C = Q_A + Q_B$. Hence outlet speed:

$$v_C = \frac{Q_C}{A_C} = \frac{37\pi/5000}{9\pi/2500} = \frac{37}{18} \text{ m s}^{-1}.$$

$$Q_A = \frac{\pi}{200} \text{ m}^3 \text{ s}^{-1}, \quad Q_B = \frac{3\pi}{1250} \text{ m}^3 \text{ s}^{-1}, \quad v_C = \frac{37}{18} \text{ m s}^{-1}.$$

(b) Velocity vectors in component form

$$\mathbf{v}_A = 2\mathbf{i}, \quad \mathbf{v}_B = 1.5 \left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} \right) = \frac{3\sqrt{3}}{4}\mathbf{i} + \frac{3}{4}\mathbf{j}, \quad \mathbf{v}_C = \frac{37}{18}\mathbf{j}.$$

(c) Force on the junction from momentum change

Given $\dot{m} = \rho Q$:

$$\begin{aligned} \dot{m}_A &= 1000 \times \frac{\pi}{200} = 5\pi \text{ kg s}^{-1}, \\ \dot{m}_B &= 1000 \times \frac{3\pi}{1250} = \frac{12\pi}{5} \text{ kg s}^{-1}, \\ \dot{m}_C &= \dot{m}_A + \dot{m}_B = 5\pi + \frac{12\pi}{5} = \frac{37\pi}{5} \text{ kg s}^{-1}. \end{aligned}$$

Force exerted by water on the junction:

$$\mathbf{F} = \dot{m}_C \mathbf{v}_C - \dot{m}_A \mathbf{v}_A - \dot{m}_B \mathbf{v}_B.$$

Compute each term:

$$\begin{aligned}\dot{m}_C \mathbf{v}_C &= \frac{37\pi}{5} \cdot \frac{37}{18} \mathbf{j} = \frac{1369\pi}{90} \mathbf{j}, \\ \dot{m}_A \mathbf{v}_A &= 5\pi \cdot 2 \mathbf{i} = 10\pi \mathbf{i}, \\ \dot{m}_B \mathbf{v}_B &= \frac{12\pi}{5} \left(\frac{3\sqrt{3}}{4} \mathbf{i} + \frac{3}{4} \mathbf{j} \right) = \frac{9\sqrt{3}}{5} \pi \mathbf{i} + \frac{9}{5} \pi \mathbf{j}.\end{aligned}$$

Hence

$$\mathbf{F} = \left(0 - 10\pi - \frac{9\sqrt{3}}{5} \pi \right) \mathbf{i} + \left(\frac{1369\pi}{90} - \frac{9}{5} \pi \right) \mathbf{j} = -\pi \left(10 + \frac{9\sqrt{3}}{5} \right) \mathbf{i} + \pi \left(\frac{1369}{90} - \frac{162}{90} \right) \mathbf{j}.$$

Simplify:

$$\mathbf{F} = -\frac{\pi}{5} (50 + 9\sqrt{3}) \mathbf{i} + \frac{1207\pi}{90} \mathbf{j}.$$

$$F_x \approx -41.21 \text{ N},$$

$$F_y \approx 42.13 \text{ N}.$$

Magnitude:

$$F = \sqrt{(-41.21)^2 + (42.13)^2} \approx \sqrt{3473} = 58.9 \text{ N}.$$

Angle θ with positive x -axis (since $F_x < 0$, $F_y > 0$):

$$\theta = 180^\circ - \arctan\left(\frac{42.13}{41.21}\right) \approx 180^\circ - 45.6^\circ = 134.4^\circ.$$

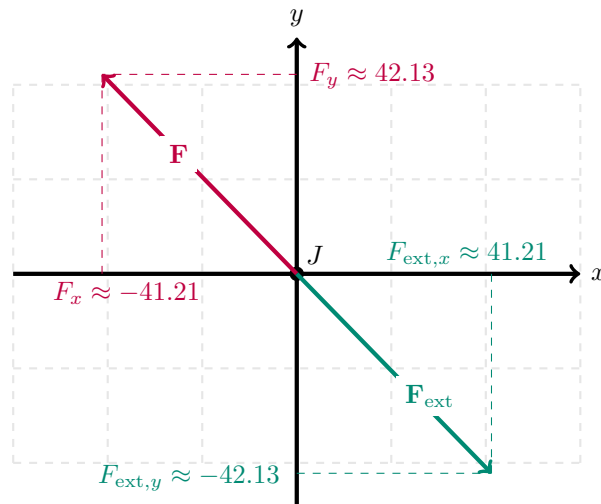
$$\boxed{\mathbf{F} \approx -41.21 \mathbf{i} + 42.13 \mathbf{j} \text{ N}, \quad F \approx 58.9 \text{ N}, \quad \theta \approx 134.4^\circ.}$$

(d) External force to hold junction

The force exerted by the water on the junction is \mathbf{F} . By Newton's third law, the junction exerts an equal and opposite force on the water. To keep the junction stationary, an external force \mathbf{F}_{ext} must balance the water force, so

$$\mathbf{F}_{\text{ext}} = -\mathbf{F} = \frac{\pi}{5} (50 + 9\sqrt{3}) \mathbf{i} - \frac{1207\pi}{90} \mathbf{j}.$$

Numerically, $\mathbf{F}_{\text{ext}} \approx 41.21 \mathbf{i} - 42.13 \mathbf{j} \text{ N}$.



The external force is the reaction that must be applied to the junction (e.g., by supports) to prevent it from accelerating.

6. A game developer is programming a 2D top-down space game. The player's spaceship is modelled as a point with position, velocity, and acceleration vectors.

At a certain game-loop update, the ship has:

- Position: $\mathbf{p} = (120\mathbf{i} + 80\mathbf{j})$ pixels
- Velocity: $\mathbf{v} = (15\mathbf{i} - 5\mathbf{j})$ pixels per frame

The game engine applies an acceleration vector $\mathbf{a} = (0.6\mathbf{i} + 0.8\mathbf{j})$ pixels per frame² for a duration of 10 frames.

- (a) Write the displacement of the ship after 10 frames if no acceleration were applied.
- (b) Including the constant acceleration \mathbf{a} , use the equation of motion

$$\mathbf{s} = \mathbf{v}t + \frac{1}{2}\mathbf{a}t^2$$

to find the total displacement over the 10 frames.

- (c) Hence find the ship's new position vector \mathbf{p}_{new} .
- (d) The game needs to check whether the ship is pointing within 30° of a target direction \mathbf{d} , given by $\mathbf{d} = 0.8\mathbf{i} + 0.6\mathbf{j}$. The ship's current direction is the unit vector in the direction of \mathbf{v} . Compute the angle between the ship's direction and \mathbf{d} using the dot product. Is the ship within the 30° tolerance?
- (e) An asteroid is centred at $\mathbf{q} = (200\mathbf{i} + 150\mathbf{j})$ pixels with a collision radius $R = 40$ pixels.
 - (i) Compute the distance from the ship's new position \mathbf{p}_{new} to the asteroid centre \mathbf{q} .
 - (ii) Compute the distance from the ship's original position \mathbf{p} to \mathbf{q} .
 - (iii) If both distances are greater than R , can we be sure that the ship did not collide with the asteroid during its movement? Give a brief reason.

Solution:

(a) Displacement without acceleration

$$\mathbf{s}_0 = \mathbf{v}t = (15\mathbf{i} - 5\mathbf{j}) \times 10 = (150\mathbf{i} - 50\mathbf{j}) \text{ pixels.}$$

$$\boxed{\mathbf{s}_0 = 150\mathbf{i} - 50\mathbf{j}}$$

(b) Total displacement with acceleration

Using $\mathbf{s} = \mathbf{v}t + \frac{1}{2}\mathbf{a}t^2$:

$$\frac{1}{2}\mathbf{a}t^2 = \frac{1}{2}(0.6\mathbf{i} + 0.8\mathbf{j}) \times 100 = (30\mathbf{i} + 40\mathbf{j}) \text{ pixels.}$$

Hence

$$\mathbf{s} = (150\mathbf{i} - 50\mathbf{j}) + (30\mathbf{i} + 40\mathbf{j}) = (180\mathbf{i} - 10\mathbf{j}) \text{ pixels.}$$

$$\boxed{\mathbf{s} = 180\mathbf{i} - 10\mathbf{j}}$$

(c) New position vector

$$\mathbf{p}_{\text{new}} = \mathbf{p} + \mathbf{s} = (120 + 180)\mathbf{i} + (80 - 10)\mathbf{j} = (300\mathbf{i} + 70\mathbf{j}) \text{ pixels.}$$

$$\boxed{\mathbf{p}_{\text{new}} = 300\mathbf{i} + 70\mathbf{j}}$$

(d) Angle between ship's direction and target direction

Ship's velocity direction: unit vector $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$.

$$|\mathbf{v}| = \sqrt{15^2 + (-5)^2} = 5\sqrt{10} \approx 15.811,$$

$$\mathbf{u} = \left(\frac{15}{5\sqrt{10}}, \frac{-5}{5\sqrt{10}} \right) = \left(\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}} \right).$$

Target direction $\mathbf{d} = 0.8\mathbf{i} + 0.6\mathbf{j}$ is a unit vector ($0.8^2 + 0.6^2 = 1$).

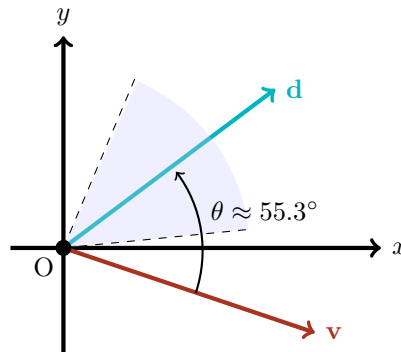
Dot product:

$$\mathbf{u} \cdot \mathbf{d} = \frac{3}{\sqrt{10}} \times 0.8 + \left(-\frac{1}{\sqrt{10}}\right) \times 0.6 = \frac{1.8}{\sqrt{10}} \approx 0.5692.$$

Angle $\theta = \arccos(0.5692) \approx 55.3^\circ$.

Since $55.3^\circ > 30^\circ$, the ship is **not** within the 30° tolerance.

$$\theta \approx 55.3^\circ, \text{ not within tolerance.}$$



(e) Asteroid collision check

Asteroid centre $\mathbf{q} = (200\mathbf{i} + 150\mathbf{j})$, radius $R = 40$ pixels.

(i) Distance from new position:

$$\mathbf{p}_{\text{new}} - \mathbf{q} = (300 - 200)\mathbf{i} + (70 - 150)\mathbf{j} = (100\mathbf{i} - 80\mathbf{j}),$$

$$d_{\text{new}} = \sqrt{100^2 + (-80)^2} \approx 128.1 \text{ pixels.}$$

(ii) Distance from original position:

$$\mathbf{p} - \mathbf{q} = (120 - 200)\mathbf{i} + (80 - 150)\mathbf{j} = (-80\mathbf{i} - 70\mathbf{j}),$$

$$d_{\text{old}} = \sqrt{(-80)^2 + (-70)^2} \approx 106.3 \text{ pixels.}$$

(iii) Both distances are greater than R . However, this does not guarantee that the ship avoided collision during its motion. The ship could have passed through the circle if the straight-line path from \mathbf{p} to \mathbf{p}_{new} comes within R of \mathbf{q} . A full collision check would require verifying that the minimum distance from the line segment to \mathbf{q} exceeds R . Hence we cannot be sure.

$$d_{\text{old}} \approx 106.3, d_{\text{new}} \approx 128.1, \text{ but not sufficient for certainty.}$$

7. In a crystal lattice analysis, the positions of atoms can be described as the sum of integer multiples of the (so-called) basis vectors \mathbf{a} and \mathbf{b} . Consider a two-dimensional hexagonal close-packed structure with basis vectors:

$$\mathbf{a} = a \left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} \right), \quad \mathbf{b} = a \left(-\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} \right)$$

where $a = 2.5 \text{ \AA}$ (angstroms).

- Show that the angle between \mathbf{a} and \mathbf{b} is 120° .
- An atom is located at position $\mathbf{r} = 2\mathbf{a} + 3\mathbf{b}$. Express this position in terms of \mathbf{i} and \mathbf{j} .
- Calculate the distance between an atom at the origin and an atom at position $\mathbf{r} = m\mathbf{a} + n\mathbf{b}$, giving your answer in terms of m , n , and a .
- For the specific case $m = 2$, $n = 3$, calculate the actual distance in angstroms.

Solution:

(a) Angle between \mathbf{a} and \mathbf{b}

Compute the dot product:

$$\mathbf{a} \cdot \mathbf{b} = a^2 \left[\frac{\sqrt{3}}{2} \cdot \left(-\frac{\sqrt{3}}{2} \right) + \frac{1}{2} \cdot \frac{1}{2} \right] = a^2 \left(-\frac{3}{4} + \frac{1}{4} \right) = -\frac{a^2}{2}.$$

Magnitudes:

$$|\mathbf{a}| = a \sqrt{\left(\frac{\sqrt{3}}{2} \right)^2 + \left(\frac{1}{2} \right)^2} = a \sqrt{\frac{3}{4} + \frac{1}{4}} = a, \quad |\mathbf{b}| = a.$$

Thus

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = -\frac{1}{2} \implies \theta = 120^\circ.$$

(b) Position $\mathbf{r} = 2\mathbf{a} + 3\mathbf{b}$ in \mathbf{i}, \mathbf{j} components

$$2\mathbf{a} = 2a \left(\frac{\sqrt{3}}{2} \mathbf{i} + \frac{1}{2} \mathbf{j} \right) = a(\sqrt{3} \mathbf{i} + \mathbf{j}),$$

$$3\mathbf{b} = 3a \left(-\frac{\sqrt{3}}{2} \mathbf{i} + \frac{1}{2} \mathbf{j} \right) = a \left(-\frac{3\sqrt{3}}{2} \mathbf{i} + \frac{3}{2} \mathbf{j} \right).$$

Adding:

$$\mathbf{r} = a \left[\left(\sqrt{3} - \frac{3\sqrt{3}}{2} \right) \mathbf{i} + \left(1 + \frac{3}{2} \right) \mathbf{j} \right] = a \left(-\frac{\sqrt{3}}{2} \mathbf{i} + \frac{5}{2} \mathbf{j} \right) = \frac{a}{2} (-\sqrt{3} \mathbf{i} + 5 \mathbf{j}).$$

(c) Distance between origin and $\mathbf{r} = m\mathbf{a} + n\mathbf{b}$

The squared distance:

$$|\mathbf{r}|^2 = (m\mathbf{a} + n\mathbf{b}) \cdot (m\mathbf{a} + n\mathbf{b}) = m^2|\mathbf{a}|^2 + n^2|\mathbf{b}|^2 + 2mn \mathbf{a} \cdot \mathbf{b}.$$

Using $|\mathbf{a}| = |\mathbf{b}| = a$ and $\mathbf{a} \cdot \mathbf{b} = -a^2/2$:

$$|\mathbf{r}|^2 = a^2 (m^2 + n^2 - mn).$$

Hence the distance is

$$\boxed{d = a\sqrt{m^2 + n^2 - mn}}.$$

(d) Distance for $m = 2$, $n = 3$ with $a = 2.5 \text{ \AA}$

$$d = 2.5\sqrt{2^2 + 3^2 - 2(3)} = 2.5\sqrt{4 + 9 - 6} = 2.5\sqrt{7}.$$

$$\boxed{d \approx 6.61 \text{ \AA}}.$$

8. A quality engineer checks a machined bracket using a coordinate measuring machine. The ideal design specifies two critical features:

- Hole A should be at position $\mathbf{a} = 20\mathbf{i} + 15\mathbf{j}$ mm from datum O .
- Hole B should be at position $\mathbf{b} = 55\mathbf{i} + 40\mathbf{j}$ mm from O .

The actual measured positions after machining are:

$$\mathbf{a}' = 20.2\mathbf{i} + 15.1\mathbf{j} \text{ mm}, \quad \mathbf{b}' = 54.8\mathbf{i} + 40.3\mathbf{j} \text{ mm}.$$

(a) Calculate the deviation vectors for each hole:

$$\mathbf{d}_A = \mathbf{a}' - \mathbf{a}, \quad \mathbf{d}_B = \mathbf{b}' - \mathbf{b}.$$

Find the magnitude of each deviation.

(b) According to the engineering drawing, the position tolerance for each hole is 0.3 mm. Are

the measured hole positions within tolerance? Justify.

- (c) The distance between holes is also critical. Calculate the ideal distance $|\mathbf{b} - \mathbf{a}|$ and the actual distance $|\mathbf{b}' - \mathbf{a}'|$. If the allowed distance tolerance is ± 0.5 mm, is the part acceptable on this feature?
- (d) The engineer also needs to check the angle of line AB relative to the horizontal. Calculate the ideal angle θ (between $\mathbf{b} - \mathbf{a}$ and \mathbf{i}) and the actual angle θ' (between $\mathbf{b}' - \mathbf{a}'$ and \mathbf{i}). If the angular tolerance is $\pm 0.5^\circ$, is the angle within specification?
- (e) Statistical process control uses the average deviation vector to detect systematic machine errors. Compute the mean deviation vector:

$$\bar{\mathbf{d}} = \frac{1}{2}(\mathbf{d}_A + \mathbf{d}_B).$$

Interpret the direction of $\bar{\mathbf{d}}$ in terms of machine misalignment.

Solution:

(a) Deviation vectors and magnitudes

$$\mathbf{d}_A = \mathbf{a}' - \mathbf{a} = (20.2 - 20)\mathbf{i} + (15.1 - 15)\mathbf{j} = 0.2\mathbf{i} + 0.1\mathbf{j} \text{ mm,}$$

$$|\mathbf{d}_A| = \sqrt{0.2^2 + 0.1^2} \approx 0.224 \text{ mm.}$$

$$\mathbf{d}_B = \mathbf{b}' - \mathbf{b} = (54.8 - 55)\mathbf{i} + (40.3 - 40)\mathbf{j} = -0.2\mathbf{i} + 0.3\mathbf{j} \text{ mm,}$$

$$|\mathbf{d}_B| = \sqrt{(-0.2)^2 + 0.3^2} \approx 0.361 \text{ mm.}$$

$$\boxed{|\mathbf{d}_A| \approx 0.224 \text{ mm, } |\mathbf{d}_B| \approx 0.361 \text{ mm}.}$$

(b) Tolerance check

Position tolerance is 0.3 mm. Hole A: $0.224 < 0.3 \rightarrow$ within tolerance. Hole B: $0.361 > 0.3 \rightarrow$ exceeds tolerance.

$$\boxed{\text{Hole A OK, Hole B NOT OK}.}$$

(c) Distance between holes

Ideal vector $\mathbf{b} - \mathbf{a} = (55 - 20)\mathbf{i} + (40 - 15)\mathbf{j} = 35\mathbf{i} + 25\mathbf{j}$ mm. Ideal distance:

$$d_{\text{ideal}} = \sqrt{35^2 + 25^2} \approx 43.012 \text{ mm.}$$

Actual vector $\mathbf{b}' - \mathbf{a}' = (54.8 - 20.2)\mathbf{i} + (40.3 - 15.1)\mathbf{j} = 34.6\mathbf{i} + 25.2\mathbf{j}$ mm. Actual distance:

$$d_{\text{actual}} = \sqrt{34.6^2 + 25.2^2} \approx 42.804 \text{ mm.}$$

Difference $d_{\text{actual}} - d_{\text{ideal}} \approx -0.208$ mm. Allowed tolerance ± 0.5 mm \rightarrow within specification.

$$\boxed{d_{\text{ideal}} \approx 43.01 \text{ mm, } d_{\text{actual}} \approx 42.80 \text{ mm, acceptable}.}$$

(d) Angle of line AB relative to horizontal

Ideal angle θ (from \mathbf{i}):

$$\tan \theta = \frac{25}{35} \approx 0.7142857 \quad \Rightarrow \quad \theta \approx \arctan(0.7142857) \approx 35.54^\circ.$$

Actual angle θ' :

$$\tan \theta' = \frac{25.2}{34.6} \approx 0.728323 \quad \Rightarrow \quad \theta' \approx \arctan(0.728323) \approx 36.07^\circ.$$

Difference $\theta' - \theta \approx 0.53^\circ$. Angular tolerance $\pm 0.5^\circ \rightarrow$ slightly exceeded.

$$\boxed{\theta \approx 35.54^\circ, \theta' \approx 36.07^\circ, \text{ NOT within tolerance}.}$$

(e) Mean deviation vector

$$\bar{\mathbf{d}} = \frac{1}{2}(\mathbf{d}_A + \mathbf{d}_B) = \frac{1}{2}[(0.2 - 0.2)\mathbf{i} + (0.1 + 0.3)\mathbf{j}] = \frac{1}{2}(0\mathbf{i} + 0.4\mathbf{j}) = 0.2\mathbf{j} \text{ mm.}$$

The mean deviation is purely in the positive \mathbf{j} direction (vertical). This indicates a systematic shift of the machined features upward by about 0.2 mm on average, suggesting that the machine's y-axis may be misaligned or offset.

$$\boxed{\bar{\mathbf{d}} = 0.2\mathbf{j} \text{ mm.}}$$

9. During re-entry, a spacecraft's velocity vector relative to the atmosphere determines the heating and loads on the heat shield. At a certain altitude, the velocity vector is measured as:

$$\mathbf{v} = 6.8\mathbf{i} - 0.45\mathbf{j} \text{ (km s}^{-1}\text{)}$$

where \mathbf{i} is parallel to the Earth's surface at that point (notionally horizontal), and \mathbf{j} is vertically upward.

- Calculate the magnitude of the velocity (the speed).
- Find the angle below the horizontal that the velocity vector makes. This is called the flight-path angle, ϕ .
- The horizontal component of velocity affects how far the spacecraft can glide. What percentage of the total speed is in the horizontal direction?
- The spacecraft's heat shield is designed to withstand a maximum re-entry angle of 4.5° . Is the current flight-path angle within the safe limit?
- If the flight-path angle is too steep, the spacecraft experiences higher heating; if it is too shallow, it may "skip" off the atmosphere. Explain in one or two sentences why a steeper angle increases the heating rate.

Mission control considers two possible adjustments to the velocity vector to reduce the flight-path angle to exactly 3.0° while keeping the speed approximately the same. One option is to increase the horizontal component; the other is to decrease the vertical component.

- Which adjustment would require a smaller change in the vector? Justify your answer with a brief vector diagram or explanation.

Solution:
(a) Magnitude of velocity (speed)

$$v = |\mathbf{v}| = \sqrt{(6.8)^2 + (-0.45)^2} = \sqrt{46.4425} \approx 6.815 \text{ km s}^{-1}.$$

$$\boxed{v \approx 6.82 \text{ km s}^{-1}}.$$

(b) Flight-path angle ϕ below horizontal

The angle below the horizontal satisfies

$$\tan \phi = \frac{\text{vertical downward component}}{\text{horizontal component}} = \frac{0.45}{6.8} \approx 0.066176.$$

Thus

$$\phi = \arctan(0.066176) \approx 3.79^\circ.$$

$$\boxed{\phi \approx 3.79^\circ \text{ below horizontal}}.$$

(c) Percentage of speed in horizontal direction

Horizontal speed $v_x = 6.8 \text{ km s}^{-1}$. Percentage:

$$\frac{v_x}{v} \times 100\% = \frac{6.8}{6.815} \times 100\% \approx 99.78\%.$$

$$\approx 99.8\%$$

(d) Comparison with safety limit

The heat shield is designed for a maximum re-entry angle of 4.5° . Current flight-path angle $\phi \approx 3.79^\circ < 4.5^\circ$, so it is within the safe limit.

Yes, within safe limit.

(e) Why a steeper angle increases heating rate

A steeper flight-path angle means the spacecraft descends more rapidly through the atmosphere, encountering higher atmospheric density at higher speeds, which increases the rate of aerodynamic heating.

(f) Adjustment to achieve $\phi = 3.0^\circ$ with minimal change

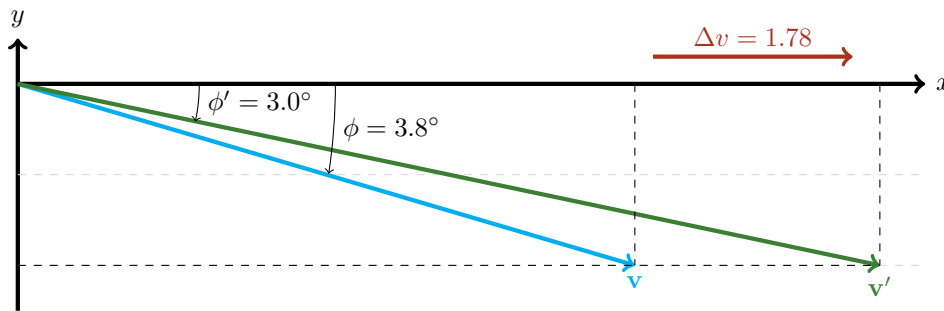
We want $\phi' = 3.0^\circ$, so let v'_x and v'_y be the new components (with v'_y positive downward magnitude).

The condition is $\frac{v'_y}{v'_x} = \tan 3.0^\circ$ and speed $v \approx 6.815 \text{ km s}^{-1}$.

- **Option 1: Increase horizontal component** (keep $v_y = 0.45$ fixed).

$$v'_x = \frac{0.45}{\tan 3.0^\circ} \approx 8.585 \text{ km s}^{-1}.$$

$$\Delta \mathbf{v} = (8.585 - 6.8) \mathbf{i} = 1.785 \mathbf{i}.$$

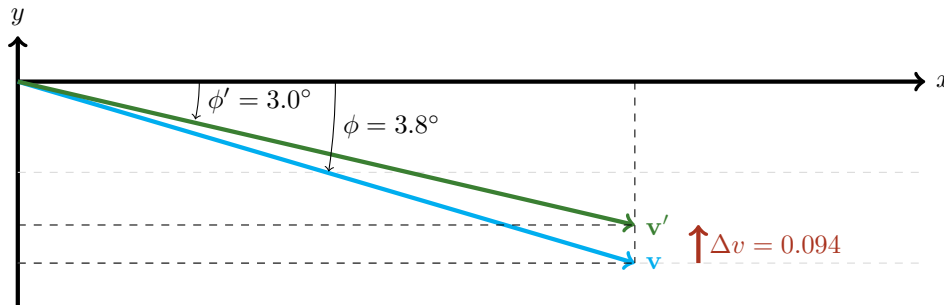


- **Option 2: Decrease vertical component** (keep $v_x = 6.8$ fixed).

$$v'_y = 6.8 \times \tan 3.0^\circ \approx 0.356 \text{ km s}^{-1}.$$

$$\Delta \mathbf{v} = (0.356 - 0.45)(-\mathbf{j}) = -0.094 \mathbf{j}$$

(i.e. downward component decreases by 0.094).



Conclusion: Decreasing the vertical component requires a much smaller change in the velocity vector (0.094 vs 1.78), making it the more efficient adjustment.