INSTITUTE OF MATHEMATICS & ITS APPLICATIONS

Vectors – Prereading

1. Scalar vs Vector Quantities

In engineering, we encounter two types of quantities:

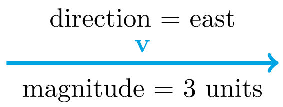
- **Scalars** are quantities that have only magnitude (size). Examples: mass (75 kg), time (10 s), temperature (20°C), speed (60 km/h).
- **Vectors** are quantities that have both magnitude and direction. Examples: velocity (60 km/h due north), force (100 N downward), displacement (5 m east), acceleration (9.8 m/s^2 downward).

Notation: In printed text, vectors are often shown in bold: $\mathbf{F}$, $\mathbf{a}$, $\mathbf{v}$. When writing by hand, we usually put a line underneath or an arrow above the letter: $\underline{v}$ or $\vec{v}$.

2. Representing Vectors Geometrically

A vector is represented geometrically by an arrow:

- The **length** of the arrow represents the magnitude.
- The **direction** of the arrow shows the direction of the vector.

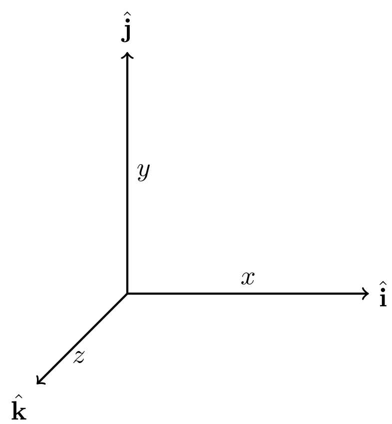


3. Unit Vectors and Basis Vectors

A **unit vector** has magnitude 1 and is often denoted with a hat: $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$. These are called **basis vectors** and define a coordinate system.
In 2D, we typically use:

- $\hat{\mathbf{i}}$ – unit vector pointing in the positive x-direction (east)
- $\hat{\mathbf{j}}$ – unit vector pointing in the positive y-direction (north)

In 3D, we add $\hat{\mathbf{k}}$ for the z-direction (into/ out of plane).



4. Component Form of Vectors

Any vector $\mathbf{v}$ can be written as a combination of basis vectors:

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}},$$

where v_x, v_y, and v_z are the **components** of the vector.

Example: $\mathbf{v} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$ means 3 units east and 4 units north.
We can also write vectors in column form:

$$\mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}.$$

5. Magnitude of a Vector

The magnitude (or length) of a vector is found using Pythagoras' theorem:

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

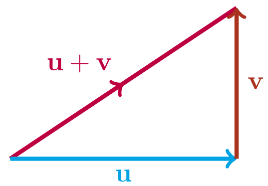
Example: For the 2D vector, $\mathbf{v} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$, $|\mathbf{v}| = \sqrt{3^2 + 4^2} = 5$.

6. Vector Addition and Subtraction

Vectors are added component-wise:

$$\mathbf{u} + \mathbf{v} = (u_x + v_x)\hat{\mathbf{i}} + (u_y + v_y)\hat{\mathbf{j}}.$$

Geometrically, we add vectors by placing them tip-to-tail.



7. Scalar Multiplication

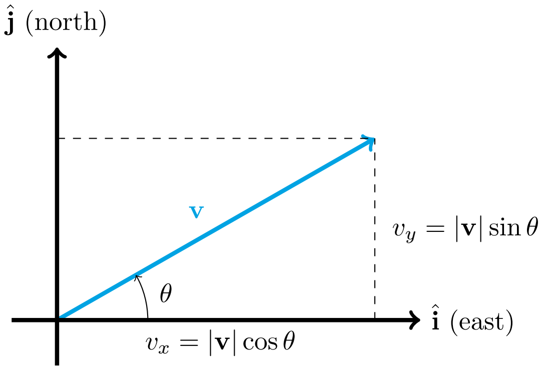
Multiplying a vector by a scalar k changes its magnitude by a factor $|k|$ and reverses direction if k is negative:

$$k\mathbf{v} = (kv_x)\hat{\mathbf{i}} + (kv_y)\hat{\mathbf{j}}.$$

Example: $2\mathbf{v}$ doubles the length; $-1\mathbf{v}$ points in the opposite direction.

8. Resolving Vectors into Components

One of the most powerful techniques in engineering is resolving a vector into its perpendicular components. This allows us to analyse the effect of a vector in different directions separately.



Given a vector $\mathbf{v}$ with magnitude $|\mathbf{v}|$ at an angle θ measured from the positive x-axis,

$$v_x = |\mathbf{v}|\cos\theta, \qquad v_y = |\mathbf{v}|\sin\theta.$$

If we know the components v_x and v_y, we can then recover the magnitude and direction using:

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2}, \qquad \tan\theta = \frac{v_y}{v_x}.$$

Choosing the right angle reference: Always be careful about which angle is given and from which direction it is measured. If the angle is measured from the positive y-axis (e.g., "north of east" problems), you may need to swap sine and cosine. In navigation problems, angles are often measured clockwise from north and so adjust accordingly!

Static Equilibrium Problems: Resolve all forces into components and set the sum in each direction to zero. Setting the sum to zero gives two scalar equations that can be solved for the unknowns.

9. Position Vectors and Displacement

A **position vector** gives the location of a point relative to the origin. The **displacement vector** from point A to point B is:

$$\overrightarrow{AB} = \mathbf{r}_B - \mathbf{r}_A.$$

10. Direction and Unit Vectors

The direction of a vector is often described by a unit vector $\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$.

Example: For velocity $\mathbf{v} = 27\hat{\mathbf{i}} - 81\hat{\mathbf{j}}$, the unit direction is

$$\hat{\mathbf{v}} = \frac{27\hat{\mathbf{i}} - 81\hat{\mathbf{j}}}{\sqrt{27^2 + (-81)^2}} = \frac{1}{\sqrt{10}}\hat{\mathbf{i}} - \frac{3\sqrt{10}}{10}\hat{\mathbf{j}}.$$

11. Dot Product (Scalar Product)

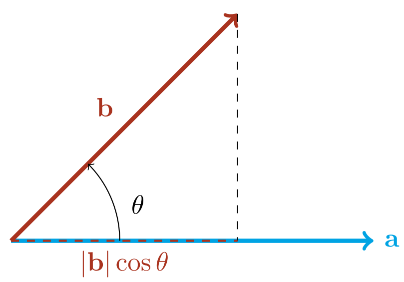
The dot product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as:

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta,$$

where θ is the angle between them. In component form:

$$\mathbf{a}\cdot\mathbf{b} = a_x b_x + a_y b_y + a_z b_z.$$

Geometric interpretation: The dot product measures how much one vector extends in the direction of another. The projection of $\mathbf{b}$ onto $\mathbf{a}$ has length $|\mathbf{b}|\cos\theta$, so $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}| \times$ (projection of $\mathbf{b}$ onto $\mathbf{a}$).



Key applications:

- Finding the angle between vectors: $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$
- Checking perpendicularity: $\mathbf{a}\cdot\mathbf{b} = 0$ means $\mathbf{a}\perp\mathbf{b}$ (the projection is zero)