INSTITUTE OF MATHEMATICS & ITS APPLICATIONS

Vectors – Problems

1. A uniform electric field $\mathbf{E}_1$ acts horizontally to the right with magnitude $200\,\mathrm{V\,m^{-1}}$, and at the same point a second field $\mathbf{E}_2$ has magnitude $150\,\mathrm{V\,m^{-1}}$ at $60°$ above the horizontal. By expressing $\mathbf{E}_1$ and $\mathbf{E}_2$ as 2D vectors in component form, find the magnitude and direction of the resultant electric field at this point.

2. During a storm, a 100 N medical-delivery drone is attempting to hover above a certain location. To prevent it from being blown to the left by high winds, two emergency tethers are quickly attached to nearby anchor points above, and on opposite sides of, the drone.

   → Tether A, anchored to the left, makes $30°$ left of vertical.
   → Tether B, anchored to the right, makes $50°$ right of vertical.
   → A violent gust exerts a steady horizontal force F to the left on the drone.

   The system is in static equilibrium.

   (a) Construct a free-body diagram for the drone and write the vector equation for static equilibrium.
   (b) By resolving forces into horizontal and vertical components, derive the two scalar equilibrium equations.
   (c) For a measured gust force of 40 N, determine the tension in each tether, T_A and T_B.
   (d) Given that both tethers have a maximum safe working load of 120 N, calculate the maximum gust force $F_{\max}$ that the system can withstand. State which tether would fail first if this limit were exceeded.
   (e) Would increasing the $30°$ left tether angle or the $50°$ right tether angle allow the system to withstand a larger maximum gust force $F_{\max}$? Provide a brief justification.

3. A communication satellite has two solar panels. Their orientation vectors (pointing normal to the panel surfaces) are given by:

$$\mathbf{a} = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k}), \quad \mathbf{b} = \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$$

   (a) Show that both $\mathbf{a}$ and $\mathbf{b}$ are unit vectors.
   (b) Calculate the angle between the two panels.
   (c) Sunlight arrives along the direction $\mathbf{s} = \mathbf{k}$. The solar power collected by a panel is proportional to the cosine of the angle between the panel's normal and the sunlight direction. Use dot products to determine the power collected by each panel.
   (d) During a manoeuvre, Panel A is rotated so that its new orientation is $\mathbf{a}' = \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{k})$. Find the new angle between Panel A' and Panel B.

4. An aircraft has an airspeed (speed relative to the air) of $300\,\mathrm{km\,h^{-1}}$. It needs to fly from Cardiff to Edinburgh, which is 600 km due north. Use a Cartesian coordinate system where the unit vector $\mathbf{i}$ points east and $\mathbf{j}$ points north.

   (a) If there is no wind, calculate the time taken for the flight.
   (b) A wind of speed $50\,\mathrm{km\,h^{-1}}$ is blowing from the west.
      (i) The pilot points the aircraft due north. Write down the air velocity vector and the wind velocity vector. Hence find the ground velocity vector and the ground speed.
      (ii) After 1 hour of flying, what is the aircraft's position relative to Cardiff? How far east has the aircraft been blown off course?
   (c) To reach Edinburgh, the pilot must adjust the heading so that the ground track is directly north. Let the heading be an angle θ measured clockwise from north.

    (i) Find the value of θ needed to cancel the effect of the wind and give a ground velocity that is due north.
    (ii) Calculate the resulting ground speed and the time for the flight.

(d) For the return flight from Edinburgh to Cardiff, the wind is still $50\,\text{km}\,\text{h}^{-1}$ from the west. The pilot wants to fly directly south (ground track due south).
    (i) Determine the required heading φ (measured clockwise from north) to achieve this.
    (ii) Compare the total time for the round trip (outbound and return) with the total time if there were no wind. Comment briefly.

5. Water flows through a horizontal pipe system with two branches meeting at a junction J. The flow is steady (not changing with time) and incompressible (water density is constant, $\rho = 1000\,\text{kg}\,\text{m}^{-3}$). At junction J:

→ Pipe A (inlet): diameter 0.1 m, water velocity $2\,\text{m}\,\text{s}^{-1}$ directed along the unit vector $\boldsymbol{\mu}_A = \mathbf{i}$
→ Pipe B (inlet): diameter 0.08 m, water velocity $1.5\,\text{m}\,\text{s}^{-1}$ directed along $\boldsymbol{\mu}_B = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$
→ Pipe C (outlet): diameter 0.12 m, water exits along $\boldsymbol{\mu}_C = \mathbf{j}$

(a) The volume flow rate Q (in $\text{m}^3\,\text{s}^{-1}$) in a pipe is given by $Q = Av$, where A is the cross-sectional area. Calculate Q_A and Q_B. For incompressible steady flow, the total inflow equals the total outflow. Use this to find the speed v_C in Pipe C.
(b) Write the velocity vectors $\mathbf{v}_A$, $\mathbf{v}_B$, $\mathbf{v}_C$ in component form.
(c) The mass flow rate is $\dot{m} = \rho Q$. The force exerted by the water on the junction is given by the momentum-change formula:

$$\mathbf{F} = \dot{m}_C\mathbf{v}_C - \dot{m}_A\mathbf{v}_A - \dot{m}_B\mathbf{v}_B$$

Calculate the force vector $\mathbf{F}$ (in N). Find its magnitude and the angle it makes with the x-axis.
(d) If the junction is anchored so that it does not move, what external force must be applied to the junction? Sketch the forces on the junction, indicating the direction of $\mathbf{F}$.

6. A game developer is programming a 2D top-down space game. The player's spaceship is modelled as a point with position, velocity, and acceleration vectors.
At a certain game-loop update, the ship has:

→ Position: $\mathbf{p} = (120\mathbf{i} + 80\mathbf{j})$ pixels
→ Velocity: $\mathbf{v} = (15\mathbf{i} - 5\mathbf{j})$ pixels per frame

The game engine applies an acceleration vector $\mathbf{a} = (0.6\mathbf{i} + 0.8\mathbf{j})$ pixels per frame^2 for a duration of 10 frames.

(a) Write the displacement of the ship after 10 frames if no acceleration were applied.
(b) Including the constant acceleration $\mathbf{a}$, use the equation of motion

$$\mathbf{s} = \mathbf{v}t + \frac{1}{2}\mathbf{a}t^2$$

to find the total displacement over the 10 frames.
(c) Hence find the ship's new position vector $\mathbf{p}_{\text{new}}$.
(d) The game needs to check whether the ship is pointing within 30° of a target direction $\mathbf{d}$, given by $\mathbf{d} = 0.8\mathbf{i} + 0.6\mathbf{j}$. The ship's current direction is the unit vector in the direction of $\mathbf{v}$. Compute the angle between the ship's direction and $\mathbf{d}$ using the dot product. Is the ship within the 30° tolerance?
(e) An asteroid is centred at $\mathbf{q} = (200\mathbf{i} + 150\mathbf{j})$ pixels with a collision radius $R = 40$ pixels.
    (i) Compute the distance from the ship's new position $\mathbf{p}_{\text{new}}$ to the asteroid centre $\mathbf{q}$.
    (ii) Compute the distance from the ship's original position $\mathbf{p}$ to $\mathbf{q}$.
    (iii) If both distances are greater than R, can we be sure that the ship did not collide with the asteroid during its movement? Give a brief reason.

7. In a crystal lattice analysis, the positions of atoms can be described as the sum of integer multiples of the (so-called) basis vectors $\mathbf{a}$ and $\mathbf{b}$. Consider a two-dimensional hexagonal close-packed structure with basis vectors:

$$\mathbf{a} = a\left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right), \quad \mathbf{b} = a\left(-\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right)$$

where $a = 2.5\,\text{Å}$ (angstroms).

(a) Show that the angle between $\mathbf{a}$ and $\mathbf{b}$ is $120°$.
(b) An atom is located at position $\mathbf{r} = 2\mathbf{a} + 3\mathbf{b}$. Express this position in terms of $\mathbf{i}$ and $\mathbf{j}$.
(c) Calculate the distance between an atom at the origin and an atom at position $\mathbf{r} = m\mathbf{a} + n\mathbf{b}$, giving your answer in terms of m, n, and a.
(d) For the specific case $m = 2$, $n = 3$, calculate the actual distance in angstroms.

8. A quality engineer checks a machined bracket using a coordinate measuring machine. The ideal design specifies two critical features:

→ Hole A should be at position $\mathbf{a} = 20\mathbf{i} + 15\mathbf{j}\,\text{mm}$ from datum O.
→ Hole B should be at position $\mathbf{b} = 55\mathbf{i} + 40\mathbf{j}\,\text{mm}$ from O.

The actual measured positions after machining are:

$$\mathbf{a}' = 20.2\mathbf{i} + 15.1\mathbf{j}\,\text{mm}, \quad \mathbf{b}' = 54.8\mathbf{i} + 40.3\mathbf{j}\,\text{mm}.$$

(a) Calculate the deviation vectors for each hole:

$$\mathbf{d}_A = \mathbf{a}' - \mathbf{a}, \quad \mathbf{d}_B = \mathbf{b}' - \mathbf{b}.$$

Find the magnitude of each deviation.
(b) According to the engineering drawing, the position tolerance for each hole is 0.3 mm. Are the measured hole positions within tolerance? Justify.
(c) The distance between holes is also critical. Calculate the ideal distance $|\mathbf{b}-\mathbf{a}|$ and the actual distance $|\mathbf{b}'-\mathbf{a}'|$. If the allowed distance tolerance is $\pm 0.5\,\text{mm}$, is the part acceptable on this feature?
(d) The engineer also needs to check the angle of line AB relative to the horizontal. Calculate the ideal angle θ (between $\mathbf{b} - \mathbf{a}$ and $\mathbf{i}$) and the actual angle θ' (between $\mathbf{b}' - \mathbf{a}'$ and $\mathbf{i}$). If the angular tolerance is $\pm 0.5°$, is the angle within specification?
(e) Statistical process control uses the average deviation vector to detect systematic machine errors. Compute the mean deviation vector:

$$\overline{\mathbf{d}} = \frac{1}{2}(\mathbf{d}_A + \mathbf{d}_B).$$

Interpret the direction of $\overline{\mathbf{d}}$ in terms of machine misalignment.

9. During re-entry, a spacecraft's velocity vector relative to the atmosphere determines the heating and loads on the heat shield. At a certain altitude, the velocity vector is measured as:

$$\mathbf{v} = 6.8\mathbf{i} - 0.45\mathbf{j}\,(\text{km}\,\text{s}^{-1})$$

where $\mathbf{i}$ is parallel to the Earth's surface at that point (notionally horizontal), and $\mathbf{j}$ is vertically upward.

(a) Calculate the magnitude of the velocity (the speed).
(b) Find the angle below the horizontal that the velocity vector makes. This is called the flight-path angle, ϕ.
(c) The horizontal component of velocity affects how far the spacecraft can glide. What percentage of the total speed is in the horizontal direction?
(d) The spacecraft's heat shield is designed to withstand a maximum re-entry angle of $4.5°$. Is the current flight-path angle within the safe limit?
(e) If the flight-path angle is too steep, the spacecraft experiences higher heating; if it is too shallow, it may "skip" off the atmosphere. Explain in one or two sentences why a steeper angle increases the heating rate.

Mission control considers two possible adjustments to the velocity vector to reduce the flight-path angle to exactly $3.0°$ while keeping the speed approximately the same. One option is to increase the horizontal component; the other is to decrease the vertical component.

(f) Which adjustment would require a smaller change in the vector? Justify your answer with a brief vector diagram or explanation.