INSTITUTE OF
MATHEMATICS &
ITS APPLICATIONS

Trigonometry – Solutions

1. In a surveying task for a future bridge project, a student engineer stands on one bank of a river and measures the angle of elevation to the top of a transmission tower on the opposite bank as 27°. She walks 95m directly away from the river in a straight line and now measures the angle of elevation as 18°. Assuming the ground is flat and the tower is vertical, what is the height h of the tower, to the nearest metre?

   ## Solution:

   From the first measurement:

$$\tan 27^\circ = \frac{h}{d} \quad \Longrightarrow \quad h = d\tan 27^\circ.$$

   From the second measurement, the distance is $d + 95$:

$$\tan 18^\circ = \frac{h}{d+95} \quad \Longrightarrow \quad h = (d+95)\tan 18^\circ.$$

   Equate the two expressions for h:

$$d\tan 27^\circ = (d+95)\tan 18^\circ.$$

   Solve for d:

$$\begin{aligned} d\tan 27^\circ &= d\tan 18^\circ + 95\tan 18^\circ, \\ d(\tan 27^\circ - \tan 18^\circ) &= 95\tan 18^\circ, \\ d &= \frac{95\tan 18^\circ}{\tan 27^\circ - \tan 18^\circ}. \\ d &\approx 167.2 \text{ m.} \end{aligned}$$

   Then the height is

$$h = d\tan 27^\circ \approx 85.2 \text{ m.}$$

   Rounding to the nearest metre gives

$$\boxed{h = 85 \text{ m}}.$$

2. A radio telescope dish has a parabolic shape. The cross-section through its centre is modelled by:

$$y(x) = 0.25x^2, \quad -8 \le x \le 8$$

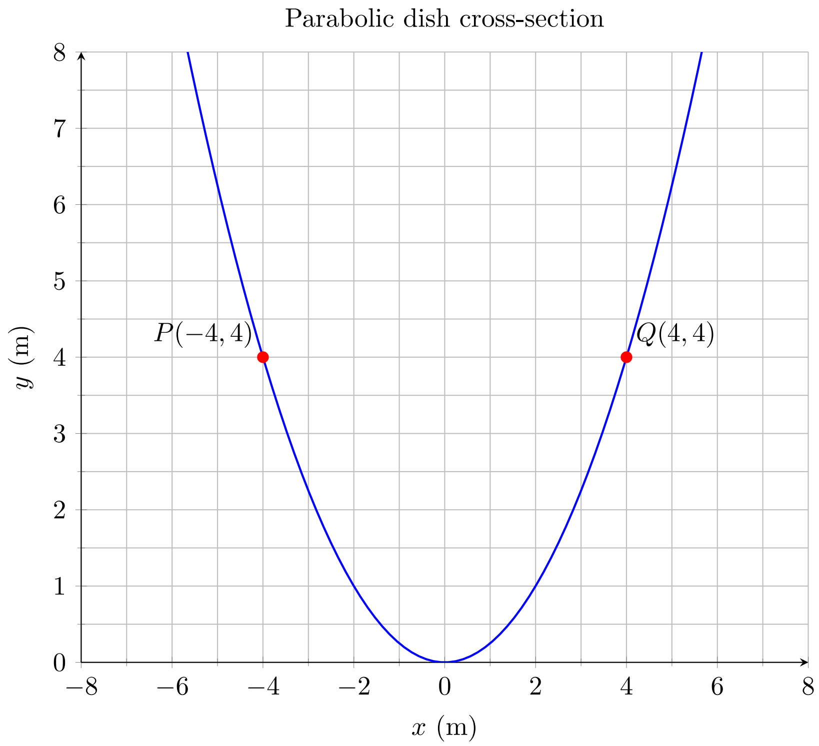
   where x and y are in metres. The dish is supported by an external framework that connects to the dish at two points, P and Q, located at $x = -4$ and $x = 4$ respectively. In addition, an internal strut runs in a straight line from P to Q.

   (a) Sketch the function $y(x)$ in the domain stated and label the points P and Q with their coordinates.
   (b) Find the equation of the internal strut PQ.
   (c) The angle θ between the dish's tangent at P and the internal strut PQ determines the stress on the mounting. Find θ in degrees to one decimal place.

(d) To reduce stress, engineers wish to adjust the external framework so that it is perpendicular to the dish at both attachment points. What would the gradient of the new external support need to be at P?

Solution:

(a) Sketch and coordinates



(b) Equation of straight support strut PQ

Both points have the same y-coordinate (4), so the strut is horizontal:

$$\boxed{y = 4}.$$

(c) Angle θ between tangent at P and strut PQ

The derivative is $\frac{dy}{dx} = 0.5x$. At $x = -4$, the tangent slope is

$$m_t = 0.5(-4) = -2.$$

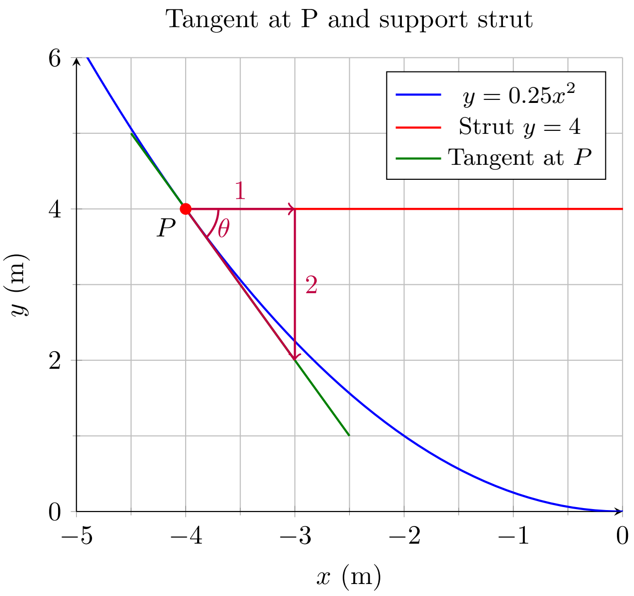
The strut slope is $m_s = 0$. The angle θ between two lines with slopes m_1 and m_2 satisfies

$$\tan\theta = \left|\frac{m_t - m_s}{1 + m_t m_s}\right| = \left|\frac{-2 - 0}{1 + (-2)(0)}\right| = 2.$$

Thus $\theta = \arctan(2) \approx 63.4349°$. To one decimal place:

$$\boxed{\theta \approx 63.4°}.$$

Alternatively, since the tangent has gradient -2, it falls 2 units vertically for every 1 unit horizontally. The angle it makes with the horizontal satisfies $\tan\theta = \frac{2}{1} = 2$, giving the same result.



(d) Gradient for a support perpendicular at P

If the external framework is perpendicular to the dish at P, its slope must be the negative reciprocal of the tangent slope at P:

$$m_\perp = -\frac{1}{m_t} = -\frac{1}{-2} = \frac{1}{2}.$$

Thus the required gradient is

$$\boxed{m_\perp = 0.5}.$$

3. During a system overload, a machine component was violently ejected from its housing. Its subsequent parabolic trajectory, starting from the ejection point $(0, 0)$, is modelled by:

$$x(t) = 12\cos(15^\circ)\,t, \qquad y(t) = 12\sin(15^\circ)\,t - 4.9t^2$$

where x is the horizontal distance in metres, y is the height in metres, and t is the time in seconds.

(a) Compare these equations to the standard SUVAT forms for projectile motion. State the component's initial speed and the ejection angle relative to the horizontal. Briefly explain why the $y(t)$ equation contains a quadratic term while the $x(t)$ equation does not.

(b) Eliminate the parameter t from $x(t)$ and $y(t)$ to show that the trajectory of the component follows the path:

$$y = \frac{x}{\tan(15^\circ)} - \frac{4.9x^2}{144\cos^2(15^\circ)}$$

(c) Using the exact trigonometric values:

$$\tan(15^\circ) = 2 - \sqrt{3} \quad \text{and} \quad \cos^2(15^\circ) = \frac{2+\sqrt{3}}{4}$$

rewrite the trajectory equation from part (b), without any trigonometric functions, such that all denominators are integers.

(d) Using the simplified exact equation from part (c), determine the horizontal distance from the machine, measured in metres to two decimal places, at which the component strikes the ground.

Solution:

(a) Comparison with SUVAT equations

The standard SUVAT equations for projectile motion (with initial speed u, launch angle θ, and gravity $g = 9.8\,\mathrm{m/s^2}$) are

$$x(t) = u\cos\theta\, t, \qquad y(t) = u\sin\theta\, t - \tfrac{1}{2}gt^2.$$

Comparing, we identify $u = 12\,\mathrm{m/s}$ and $\theta = 15°$. The $x(t)$ equation is linear because there is no horizontal acceleration (air resistance neglected). The $y(t)$ equation contains a quadratic term $-\frac{1}{2}gt^2$ due to the constant downward acceleration of gravity.

(b) Eliminate t to obtain the trajectory

From $x(t)$: $t = \dfrac{x}{12\cos 15°}$. Substitute into $y(t)$:

$$y = 12\sin 15° \cdot \frac{x}{12\cos 15°} - 4.9\left(\frac{x}{12\cos 15°}\right)^2 = x\tan 15° - \frac{4.9\,x^2}{144\cos^2 15°}.$$

Thus

$$\boxed{y = x\tan 15° - \frac{4.9\,x^2}{144\cos^2 15°}}.$$

(c) Exact trigonometric substitution

Given $\tan 15° = 2-\sqrt{3}$ and $\cos^2 15° = \dfrac{2+\sqrt{3}}{4}$. Substitute:

$$y = x(2-\sqrt{3}) - \frac{4.9\,x^2}{144\cdot\frac{2+\sqrt{3}}{4}} = x(2-\sqrt{3}) - \frac{19.6\,x^2}{144(2+\sqrt{3})}.$$

$$y = x(2-\sqrt{3}) - \frac{49}{360}\left[\frac{x^2}{2+\sqrt{3}}\right].$$

Rationalise the second term by multiplying numerator and denominator by $2-\sqrt{3}$:

$$\frac{x^2}{2+\sqrt{3}} = x^2\left[\frac{2-\sqrt{3}}{(2+\sqrt{3})(2-\sqrt{3})}\right] = x^2(2-\sqrt{3}),$$

Therefore,

$$y = x(2-\sqrt{3}) - \frac{49}{360}\,x^2(2-\sqrt{3}) = (2-\sqrt{3})\left(x - \frac{49}{360}x^2\right).$$

$$\boxed{y = (2-\sqrt{3})\left(x - \frac{49}{360}x^2\right)}.$$

(d) Horizontal distance when it strikes the ground

The component hits the ground when $y = 0$. Using the exact equation:

$$(2-\sqrt{3})\left(x - \frac{49}{360}x^2\right) = 0.$$

Since $2-\sqrt{3} \neq 0$, we have $x - \frac{49}{360}x^2 = 0 \;\Rightarrow\; x\left(1 - \frac{49}{360}x\right) = 0$. The non-zero solution is

$$x = \frac{360}{49}\ \mathrm{m}.$$

$$\boxed{x = \frac{360}{49}\ \mathrm{m} \approx 7.35\ \mathrm{m}}.$$

Note: The solution $x = 0$ corresponds to the instant the component is ejected from its housing (i.e., its launch point). The non-zero solution gives the impact point!

4. An oceanographer stands on the deck of a research ship, with her eyes at a height h above sea level. The Earth is modelled as a sphere of radius R.
   (a) Draw a clear sketch showing:
      i. The centre of the Earth,

ii. The observer at height h above the surface,
iii. The point on the sea surface where the line of sight is tangent to the Earth (the horizon).

Mark the right angle at the horizon point.

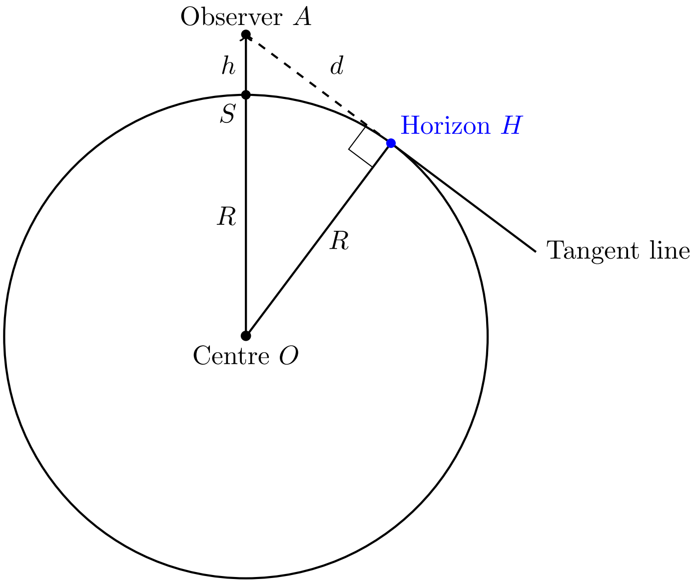
(b) Using Pythagoras' theorem on the right-angled triangle formed, derive an expression for the straight-line distance d from the observer to the horizon in terms of R and h.

(c) Given the Earth's radius $R = 6371\,\text{km}$, use graphing software to plot the horizon distance d (in km) as a function of height h (in m) for $0 \le h \le 100\,\text{m}$. State, looking from a height of $2\,\text{m}$ above sea level, how far away is the horizon (to 3 significant figures)?

(d) A coastal lighthouse needs to be visible 20 nautical miles ($\approx 37\,\text{km}$) out to sea. Calculate the minimum height of the lighthouse light, to the nearest metre above sea level, if atmospheric refraction extends the horizon by 8%.

Solution:

(a) Diagram



The diagram shows the centre O of the Earth, the observer A at height h above the surface point S, and the horizon point H where the line of sight from A is tangent to the Earth's surface. The radius OH is perpendicular to the tangent line AH, so triangle OHA is right-angled at H.

(b) Derivation of horizon distance d

In right triangle OHA:

$$OA = R + h, \quad OH = R, \quad AH = d.$$

By Pythagoras:

$$OA^2 = OH^2 + AH^2 \quad \Longrightarrow \quad (R+h)^2 = R^2 + d^2.$$

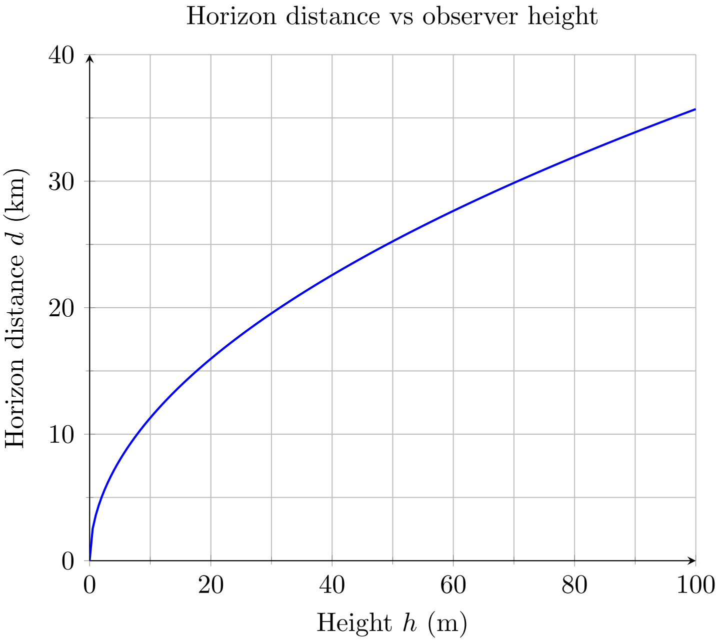
Expand:

$$R^2 + 2Rh + h^2 = R^2 + d^2 \quad \Longrightarrow \quad d^2 = 2Rh + h^2.$$

Since $h \ll R$ in typical cases, the h^2 term is often neglected, but we keep it exact. Thus

$$\boxed{d = \sqrt{2Rh + h^2}}.$$

(c) Horizon distance for $h = 2$m



Given $R = 6371$km. We must use consistent units: convert h to kilometres: $h = 0.002$km.

$$d = \sqrt{2(6371)(0.002) + (0.002)^2} \approx 5.048 \text{ km}.$$

To three significant figures, $d \approx 5.05$km.

$$\boxed{d \approx 5.05 \text{ km}}.$$

If using the approximation $d \approx \sqrt{2Rh}$, we get $d \approx 5.05$ as well, because the h^2 term is negligible.

(d) Lighthouse height calculation

Atmospheric refraction extends the horizon distance by 8%. This means that for a given height, the actual visible distance is 8% greater than the geometric distance calculated from Pythagoras. Hence, to achieve a visible distance of 37 km, the geometric line-of-sight distance required is,

$$d_{\text{geo}} = \frac{37}{1.08} \approx 34.259 \text{ km}.$$

Then using $d_{\text{geo}} = \sqrt{2Rh + h^2}$ (or approximation), we get

$$h \approx \frac{d_{\text{geo}}^2}{2R} = \frac{(34.259)^2}{2 \times 6371} \approx 0.0921 \text{ km} = 92.1 \text{ m}.$$

So, to be visible at 20 nautical miles (once refraction is included), the lighthouse light must be at a height of about,

$$\boxed{h \approx 92.1 \text{ m above sea level}}.$$

5. In a lens coating facility, engineers analyse optical path differences to ensure interference effects meet design specifications. For a specific anti-reflective coating, the path difference Δ between two light rays is given by:

$$\Delta = d(\sec\theta - 1)$$

where $d = 0.1\,\mu$m is the coating thickness, and θ is the angle of incidence in radians.

   (a) Using the small-angle approximation $\sec\theta \approx 1 + \frac{\theta^2}{2}$, derive a simplified expression for the path difference Δ.

   (b) For $\theta = 5°$, calculate the percentage error between the exact and approximate values of Δ.

When light passes from one medium into another, it bends according to Snell's Law:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

(c) For $n_1 = 1.0$ (air), $n_2 = 1.5$ (glass), and $\theta_1 = 3°$, find θ_2 using the small-angle approximation $\sin\theta \approx \theta$. Comment on the approximation's validity in this case.

The manufacturing division requires that for production-line alignment calculations, the small-angle approximation error for path difference must be less than 0.1% of the exact value.

(d) Write down the inequality that expresses this condition.

(e) Using numerical trial-and-error, or graphing software, determine the maximum allowable angle $\theta_{\max}$ (in degrees) that satisfies this 0.1% tolerance requirement.

Solution:

(a) Small-angle approximation

Using $\sec\theta \approx 1 + \dfrac{\theta^2}{2}$ for small θ, we obtain

$$\Delta \approx d\left(1 + \frac{\theta^2}{2} - 1\right) = \frac{d}{2}\theta^2.$$

$$\boxed{\Delta_{\text{approx}} = \frac{d}{2}\theta^2}.$$

(b) Percentage error at $\theta = 5°$

First convert to radians: $\theta = 5° = 5 \times \frac{\pi}{180} = \frac{\pi}{36} \approx 0.0872665$ rad.
Exact value:

$$\Delta_{\text{exact}} = d(\sec\theta - 1) = 0.01(10^{-6})\left(\frac{1}{\cos\theta} - 1\right).$$

$$\Delta_{\text{exact}} = 0.01(10^{-6})\left(\frac{1}{\cos 5°} - 1\right) \approx 3.8198 \times 10^{-11} \text{ m}.$$

Approximation:

$$\Delta_{\text{approx}} = \frac{d}{2}\theta^2 = \frac{0.01(10^{-6})}{2} \times (0.0872665)^2 = 3.8077 \times 10^{-11} \text{ m}.$$

Percentage error:

$$\text{Error} = \left|\frac{\Delta_{\text{exact}} - \Delta_{\text{approx}}}{\Delta_{\text{exact}}}\right| \times 100\% \approx \frac{3.8198 - 3.8077}{3.8198} \times 100\% \approx 0.317\%.$$

$$\boxed{\text{Error} \approx 0.317\%}.$$

(c) Snell's law with small-angle approximation

Snell's law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$. Given $n_1 = 1.0$, $n_2 = 1.5$, $\theta_1 = 3° = \frac{\pi}{60} \approx 0.0523599$ rad.
Using small-angle approximation $\sin\theta \approx \theta$:

$$\theta_2 \approx \frac{n_1}{n_2}\theta_1 = \frac{1.0}{1.5} \times 0.0523599 \approx 0.0349066 \text{ rad} \approx 2.0°.$$

The exact value would be $\theta_2 = \arcsin\left(\frac{n_1}{n_2}\sin\theta_1\right) = \arcsin\left(\frac{1}{1.5}\sin 3°\right) \approx 0.03490$ rad, which is essentially the same as the approximation. The approximation is valid in this case.

(d) Condition for 0.1% tolerance

The relative error must be less than $0.1\% = 0.001$:

$$\frac{|\Delta_{\text{exact}} - \Delta_{\text{approx}}|}{\Delta_{\text{exact}}} < 0.001.$$

Since $\Delta_{\text{exact}} > \Delta_{\text{approx}}$ for $\theta > 0$, this becomes

$$\frac{\sec\theta - 1 - \frac{\theta^2}{2}}{\sec\theta - 1} < 0.001.$$

(e) Maximum allowable angle $\theta_{\max}$

We solve the inequality numerically. Let the relative error function be

$$E(\theta) = \frac{\sec\theta - 1 - \frac{\theta^2}{2}}{\sec\theta - 1}.$$

From part (b), at $\theta = 5°$ the error is about 0.317%, which exceeds the 0.1% tolerance. We therefore test smaller angles.

- For $\theta = 2.5° = \dfrac{2.5\pi}{180}$ rad:

  $$\text{Error} = \frac{\sec(2.5°) - 1 - 0.5\left(\frac{2.5\pi}{180}\right)^2}{\sec(2.5°) - 1} \times 100 \approx 0.0793\%,$$

  which is below 0.1%.
- For $\theta = 3.0° = \dfrac{3\pi}{180}$ rad:

  $$\text{Error} \approx 0.1142\%,$$

  which is above 0.1%.

Thus the critical angle lies between 2.5° and 3.0°. By further refinement (e.g., testing $\theta = 2.8°$), we find that the error reaches 0.1% very close to $\theta = 2.8°$. Hence the maximum allowable angle satisfying the specification is approximately

$$\boxed{\theta_{\max} \approx 2.8°}.$$

6. A small sensor moves on a circular path of radius 1m in a horizontal plane, centred at the origin O. Its position at time t seconds is modelled by the point P with co-ordinates $(\cos\theta, \sin\theta)$, where θ is the angle (in radians) measured from the positive x-axis to the line OP. A receiver is fixed at the point A with co-ordinates $(2, 0)$.

   (a) Verify, using a suitable trigonometric identity, that the point P traverses a circle of radius 1 centred on the origin.

   (b) Show that the distance AP between the receiver and the sensor can be written as

   $$AP^2 = 5 - 4\cos\theta.$$

   (c) The signal strength S received at A is inversely proportional to the square of the distance from the sensor:

   $$S = \frac{k}{AP^2}$$

   for some constant k. The sensor is said to be in the "high-signal zone" when $S > \dfrac{k}{3}$. By expressing S in terms of $\cos\theta$, find the range of values of θ (in radians, within $0 \le \theta < 2\pi$) for which the sensor is in the high-signal zone.

   (d) Sketch the unit circle and highlight the arc(s) where the sensor is in the high-signal zone. Clearly label the principal angles on your diagram.

Solution:

(a) Verification of circular path

The coordinates of P are $(\cos\theta, \sin\theta)$. Using the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$, which holds for all θ, we have

$$(\cos\theta)^2 + (\sin\theta)^2 = 1.$$

This is exactly the equation of a circle of radius 1 centred at the origin. Hence, as θ varies, P traverses the unit circle.

(b) Distance AP squared

$$AP^2 = (2 - \cos\theta)^2 + (0 - \sin\theta)^2 = 4 - 4\cos\theta + \cos^2\theta + \sin^2\theta = 5 - 4\cos\theta.$$

$$\boxed{AP^2 = 5 - 4\cos\theta}.$$

(c) High-signal zone condition

Signal strength $S = \dfrac{k}{AP^2} = \dfrac{k}{5 - 4\cos\theta}$. The condition $S > \dfrac{k}{3}$ (with $k > 0$) becomes

$$\frac{k}{5 - 4\cos\theta} > \frac{k}{3} \quad \Longrightarrow \quad \frac{1}{5 - 4\cos\theta} > \frac{1}{3}.$$

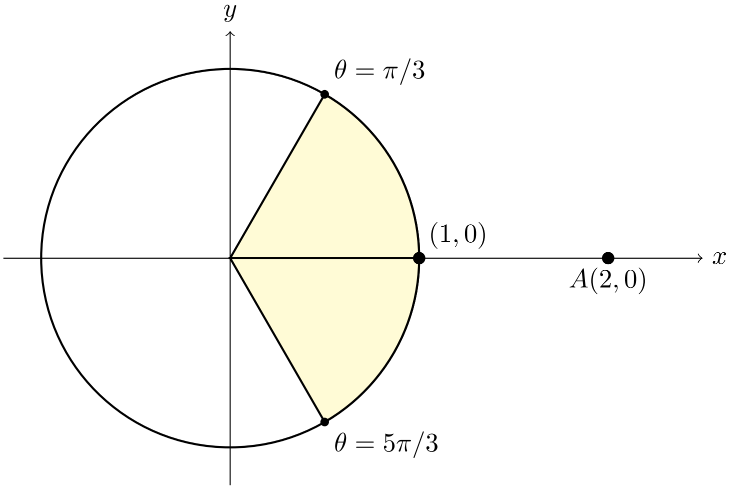
Since $5 - 4\cos\theta > 0$ (its minimum is 1), we can cross-multiply without sign change:

$$3 > 5 - 4\cos\theta \quad \Longrightarrow \quad 4\cos\theta > 2 \quad \Longrightarrow \quad \cos\theta > \frac{1}{2}.$$

Within $0 \le \theta < 2\pi$, $\cos\theta > \frac{1}{2}$ for

$$\theta \in \left(0, \frac{\pi}{3}\right) \cup \left(\frac{5\pi}{3}, 2\pi\right).$$

(d) Sketch of the unit circle with high-signal zone



Shaded arcs: $\cos\theta > \frac{1}{2}$

The diagram shows the unit circle. The two arcs (from 0 to $\pi/3$ and from $5\pi/3$ to 2π) are shaded. On these arcs the sensor's x-coordinate exceeds 0.5, so it lies closer to the receiver at $(2, 0)$ and thus receives higher signal strength.

7. In a university laboratory, a single-phase AC voltage supply is described by

$$v(t) = 170\cos(100\pi t + 30^\circ)$$

where voltage is in volts and time t is in seconds.

   (a) By plotting the function $v(t)$, state the peak (maximum) voltage and the phase angle of this supply.

The root-mean-square (RMS) voltage is the DC-equivalent voltage that produces the same heating effect in a resistor. It is given by

$$V_{\text{RMS}} = \frac{V_{\text{peak}}}{\sqrt{2}}.$$

In the remaining parts of this question, give your answers to one decimal place.

(b) Calculate the RMS voltage of this supply.

(c) On a phasor diagram, this source is represented as a vector of length 170 at an angle of 30° to the horizontal. Find the horizontal and vertical components, v_x and v_y, of this phasor.

(d) A second AC source has the same frequency, a peak voltage of 120 V, and a phase angle of $-15°$. Draw its phasor on the same diagram, and find its horizontal and vertical components.

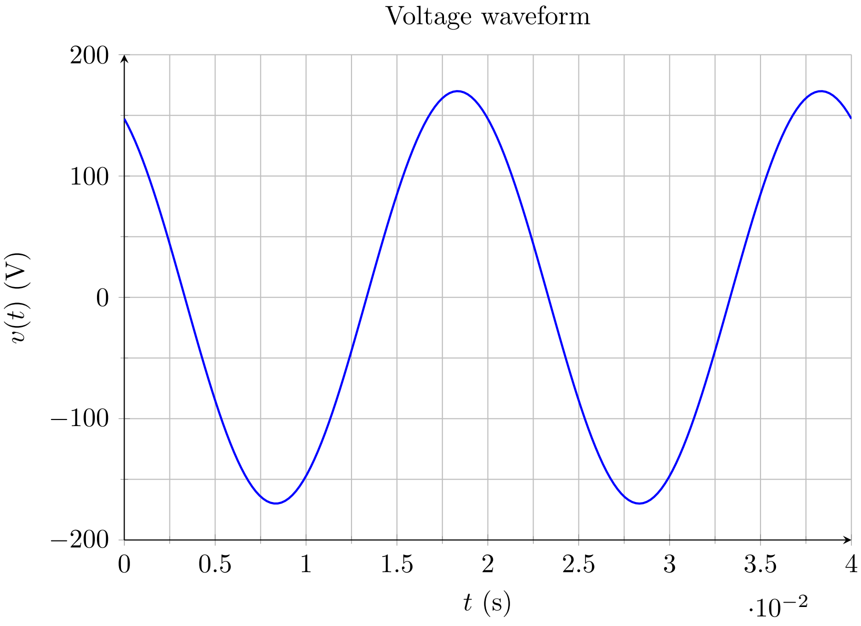
(e) If the two sources are connected in series so that their voltages add as vectors, find the magnitude of the resultant peak voltage.

(f) Convert the resultant peak voltage found in part (e) to an RMS voltage using the formula given earlier.

Solution:

(a) Peak voltage and phase angle.

From the expression, the peak voltage is the amplitude: $V_{\text{peak}} = 170$ V. The phase angle is the constant shift in the cosine argument: 30°. A sketch of $v(t)$ shows a cosine wave with amplitude 170, shifted left by 30° (or $\pi/6$ rad) relative to a pure cosine.



Thus $\boxed{V_{\text{peak}} = 170 \text{ V}, \quad \phi = 30°}$.

(b) RMS voltage

$$V_{\text{RMS}} = \frac{V_{\text{peak}}}{\sqrt{2}} = \frac{170}{\sqrt{2}} \approx 120.2 \text{ V} \quad \text{(to one decimal)}.$$

$$\boxed{V_{\text{RMS}} \approx 120.2 \text{ V}}.$$

(c) Phasor components of the first source

The phasor has length 170 at angle 30°. Its horizontal and vertical components are

$$v_x = 170\cos 30° = 170 \times \frac{\sqrt{3}}{2} \approx 147.2 \text{ V},$$

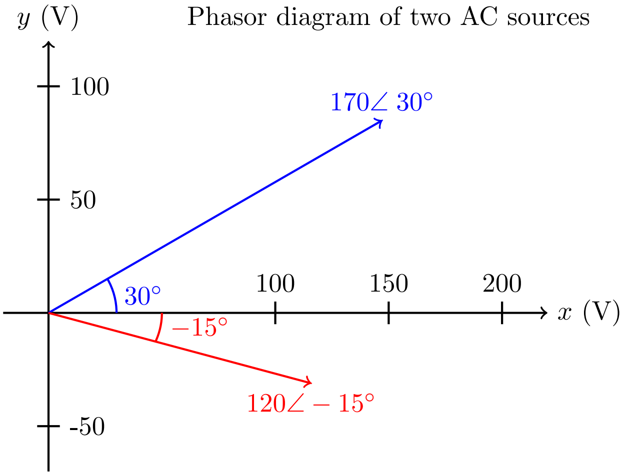
$$v_y = 170 \sin 30^\circ = 170 \times \frac{1}{2} = 85 \text{ V}.$$

$$\boxed{v_x \approx 147.2 \text{ V}, \quad v_y = 85 \text{ V}}.$$

(d) Second source components and phasor diagram
Second source: peak 120 V, phase -15°. Its components:

$$w_x = 120 \cos(-15^\circ) = 120 \cos 15^\circ, \qquad w_y = 120 \sin(-15^\circ) = -120 \sin 15^\circ.$$

$$\boxed{w_x \approx 115.9 \text{ V}, \quad w_y \approx -31.1 \text{ V}}.$$



(e) Resultant peak voltage when connected in series
The total phasor components are

$$V_x = v_x + w_x \approx 147.2 + 115.9 = 263.1 \text{ V}, \qquad V_y = v_y + w_y = 85 + (-31.1) = 53.9 \text{ V}.$$

The magnitude (peak voltage) is

$$V_{\text{peak,res}} = \sqrt{V_x^2 + V_y^2} \approx \sqrt{263.1^2 + 53.9^2} \approx 268.6 \text{ V}.$$

$$\boxed{V_{\text{peak,res}} \approx 268.6 \text{ V}}.$$

(f) RMS of resultant

$$V_{\text{RMS,res}} = \frac{V_{\text{peak,res}}}{\sqrt{2}} \approx \frac{268.6}{1.4142} \approx 190.0 \text{ V}.$$

$$\boxed{V_{\text{RMS,res}} \approx 190.0 \text{ V}}.$$

8. In a car gearbox, meshing gear teeth generate harmonic forces. The net tangential force F on a gear at angular position ϕ is the superposition of two pressure components from the driving and driven gear teeth:

$$F(\phi) = 5 \sin(\phi + 25^\circ) + 3 \sin(\phi - 20^\circ)$$

   (a) Using appropriate trigonometric identities, show that

   $$F(\phi) = R \sin(\phi + \gamma)$$

   stating expressions for the amplitude R and phase angle γ in terms of numerical values.

   (b) Calculate the amplitude R to three significant figures, and the phase shift γ to one decimal place (in degrees).

   (c) Without using a calculator, state the maximum and minimum values of $F(\phi)$, and the value(s) of ϕ at which the maximum occurs.

(d) The gearbox manufacturer adds a constant damping force of 1.2 units. Write down the new expression for the total force and state its maximum value.

(e) Sketch the graph of $F(\phi)$ for $0^\circ \le \phi \le 360^\circ$. On the same axes, sketch the graph after the damping force is added. On each sketch, clearly mark the amplitude, period, phase shift, and maximum and minimum points.

Solution:

(a) Express as a single sine function

Using the identity $\sin(\phi+\delta) = \sin\phi\cos\delta + \cos\phi\sin\delta$:

$$\begin{aligned} F(\phi) &= 5\big(\sin\phi\cos 25^\circ + \cos\phi\sin 25^\circ\big) \\ &\quad + 3\big(\sin\phi\cos 20^\circ - \cos\phi\sin 20^\circ\big) \\ &= \sin\phi\,(5\cos 25^\circ + 3\cos 20^\circ) \\ &\quad + \cos\phi\,(5\sin 25^\circ - 3\sin 20^\circ). \end{aligned}$$

Let

$$A = 5\cos 25^\circ + 3\cos 20^\circ, \qquad B = 5\sin 25^\circ - 3\sin 20^\circ.$$

Then $F(\phi) = A\sin\phi + B\cos\phi$. This can be written as $R\sin(\phi+\gamma)$ with

$$R = \sqrt{A^2+B^2}, \qquad \tan\gamma = \frac{B}{A}.$$

(b) Numerical values of R and γ

Using $\cos 25^\circ \approx 0.9063,\ \cos 20^\circ \approx 0.9397,\ \sin 25^\circ \approx 0.4226,\ \sin 20^\circ \approx 0.3420$:

$$\begin{aligned} A &\approx 5\times 0.9063 + 3\times 0.9397 = 4.5315 + 2.8191 = 7.3506, \\ B &\approx 5\times 0.4226 - 3\times 0.3420 = 2.1130 - 1.0260 = 1.0870. \end{aligned}$$

Then

$$R = \sqrt{7.3506^2 + 1.0870^2} \approx 7.430.$$

To three significant figures, $R = 7.43$.

$$\gamma = \arctan\left(\frac{B}{A}\right) = \arctan\left(\frac{1.0870}{7.3506}\right) \approx 8.42^\circ.$$

To one decimal place, $\gamma = 8.4^\circ$. Thus

$$\boxed{F(\phi) = 7.43\sin(\phi + 8.4^\circ)}.$$

(c) Maximum and minimum values

The maximum value of F is $R = 7.43$ and the minimum is -7.43.
The maximum occurs when $\sin(\phi+\gamma) = 1$, i.e.

$$\phi + 8.4^\circ = 90^\circ + 360^\circ k \quad\Rightarrow\quad \phi = 81.6^\circ + 360^\circ k.$$

For $0^\circ \le \phi < 360^\circ$, the maximum is at $\phi \approx 81.6^\circ$.

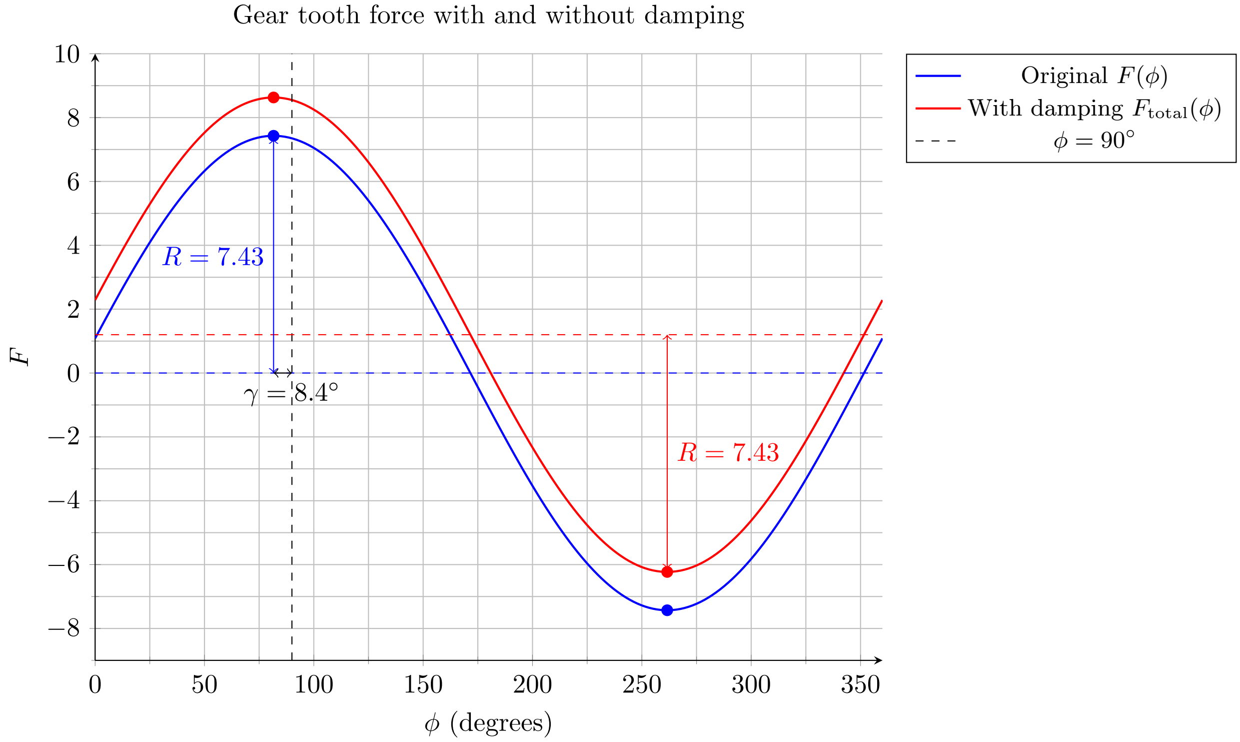
(d) Addition of constant damping force

Adding a constant damping force of 1.2 units gives

$$F_{\text{total}}(\phi) = 7.43\sin(\phi + 8.4^\circ) + 1.2.$$

The new maximum is $7.43 + 1.2 = 8.63$ and the new minimum is $-7.43 + 1.2 = -6.23$.

(e) Graphical representation



Both curves are sinusoids with period 360° and a phase shift of 8.4° to the left. The damping simply shifts the entire curve upward by 1.2 units, preserving the amplitude and phase. The marked points indicate the maxima and minima.