INSTITUTE OF MATHEMATICS & ITS APPLICATIONS

Trigonometry – Prereading

1. Angles and Their Measurement

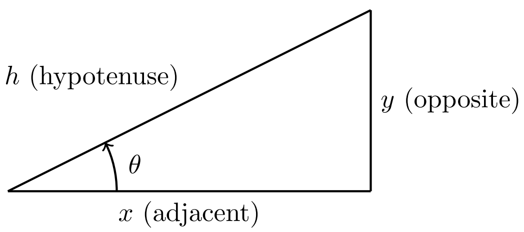
Angles can be measured in degrees or radians. One full revolution is 360° or 2π radians. The conversion between degrees and radians is fundamental:

$$180^\circ = \pi \text{ radians} \quad \Rightarrow \quad 1^\circ = \frac{\pi}{180} \text{ rad}, \quad 1 \text{ rad} = \frac{180^\circ}{\pi} \approx 57.3^\circ.$$

Why radians? In engineering mathematics, radians are the natural unit for angles because they simplify calculus. Derivatives of trigonometric functions like $\frac{d}{d\theta}\sin\theta = \cos\theta$ only hold when θ is in radians.

2. Right-Angled Triangle Trigonometry

For a right-angled triangle with hypotenuse h, opposite side y (to angle θ), and adjacent side x:



The three primary trigonometric ratios are defined as:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{h}, \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{h}, \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}.$$

Reciprocal ratios: You may also encounter:

$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}.$$

3. Pythagoras' Theorem and the Distance Formula

3.1 Pythagoras' Theorem

For any right-angled triangle with legs a and b and hypotenuse c:

$$a^2 + b^2 = c^2.$$

Example: In the triangle above, $x^2 + y^2 = h^2$. Dividing both sides by h^2 gives $\left(\frac{x}{h}\right)^2 + \left(\frac{y}{h}\right)^2 = 1$, i.e. $\cos^2\theta + \sin^2\theta = 1$ – the fundamental Pythagorean identity.

3.2 Distance Formula

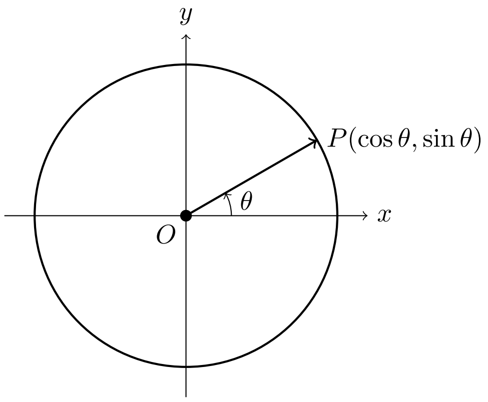
Given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the distance between them is:

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This is simply Pythagoras' theorem applied to the right triangle formed by the horizontal and vertical differences.

4. The Unit Circle and General Angles

For angles beyond 0° to 90°, the right-triangle definition fails. The unit circle provides a general definition:



A point P on the unit circle has coordinates $(\cos\theta, \sin\theta)$, where θ is the angle measured from the positive x-axis. This definition works for all angles, positive (counter-clockwise), negative (clockwise) and angles greater than 360° (multiple revolutions).

Key properties:

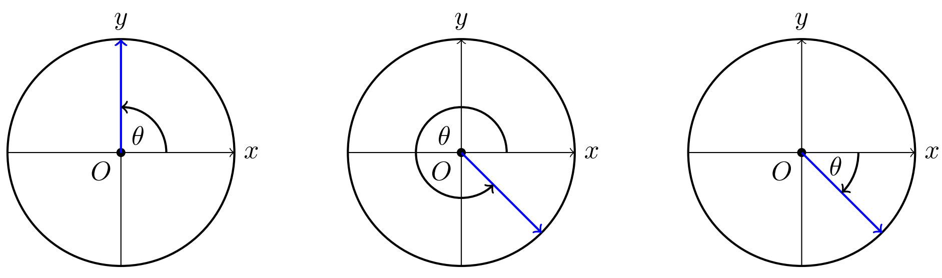
- $\cos^2\theta + \sin^2\theta = 1$ (always true – it's the circle equation)
- $\cos\theta$ and $\sin\theta$ range between -1 and 1
- The signs of $\sin\theta$ and $\cos\theta$ depend on the quadrant

Understanding radians: Radians measure angles using the unit circle. For a circle of radius r, the angle in radians is defined as

$$\theta = \frac{\text{arc length}}{\text{radius}} = \frac{l}{r}.$$

On the unit circle ($r = 1$), the angle in radians equals the arc length itself: $\theta = l$. This explains the conversion between degrees and radians:

- A full revolution is 360° and has circumference $2\pi r = 2\pi$, so $360° = 2\pi$ radians.
- Half a revolution is 180°, with arc length π, so $180° = \pi$ radians.
- A quarter revolution is 90°, with arc length $\pi/2$, so $90° = \pi/2$ radians.



$\theta = 90° = \frac{\pi}{2}$ rad $\theta = 315° = \frac{7\pi}{4}$ rad $\theta = -45° = -\frac{\pi}{4}$ rad

Thus, to convert degrees to radians, multiply by $\frac{\pi}{180}$; to convert radians to degrees, multiply by $\frac{180}{\pi}$.

5. Exact Trigonometric Values

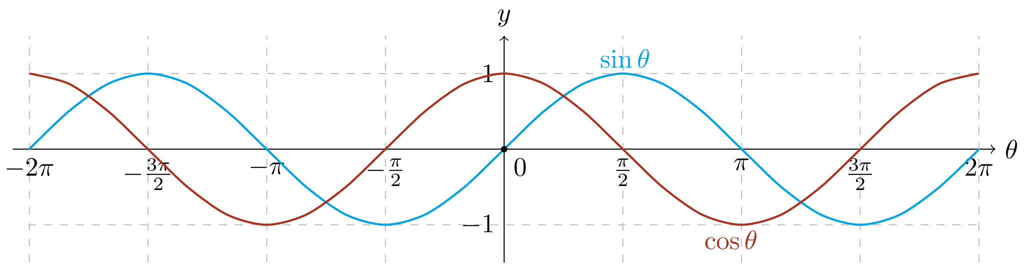
Certain angles appear frequently in engineering problems. Memorising these exact values is useful:

Angle θ	0°	30°	45°	60°	90°	180°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined	0

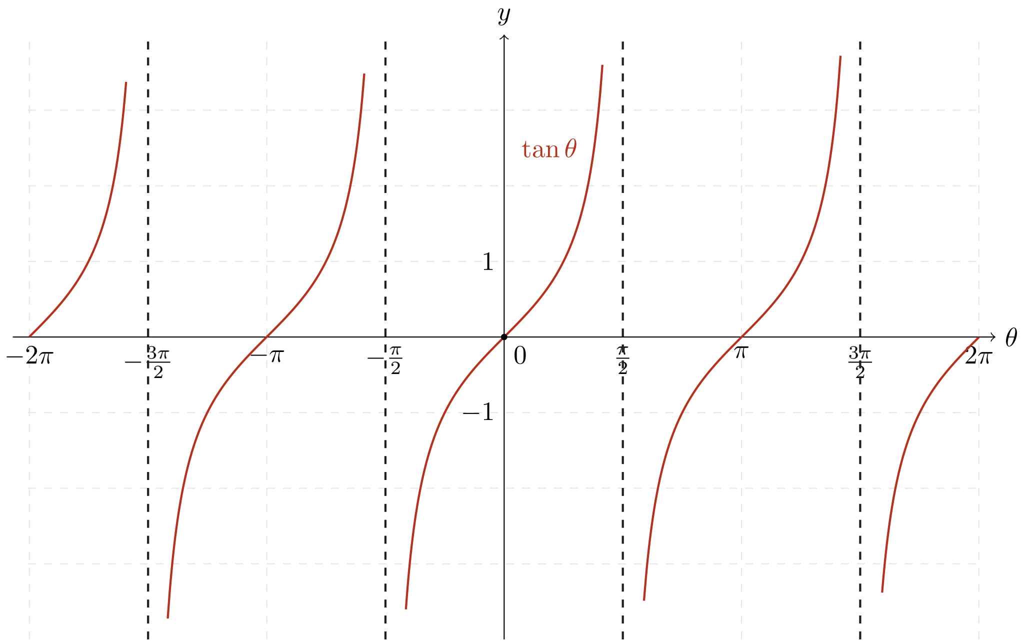
Why is $\tan 90°$ undefined? Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, when $\cos\theta = 0$ (at $\theta = 90°$ and $270°$), the denominator is zero, making the ratio undefined. On the graph of $\tan\theta$, this appears as vertical asymptotes – lines the curve approaches but never touches (see Section 6 for the graph).

6. Graphs of Trigonometric Functions

The sine and cosine functions share similar characteristics – they both repeat every 2π radians, oscillate smoothly between -1 and $+1$, and are continuous everywhere (no breaks or asymptotes).



The tangent function has a different character – it repeats every π radians, has vertical asymptotes where $\cos\theta = 0$ and ranges from $-\infty$ to $+\infty$ between asymptotes.



7. Trigonometric Identities

Identities are equations true for all angles. They are essential for simplifying expressions and solving equations.

7.1 Pythagorean Identity

$$\sin^2\theta + \cos^2\theta = 1.$$

This identity is simply Pythagoras' theorem applied to the unit circle. For any point $P(\cos\theta, \sin\theta)$ on the unit circle, the horizontal distance from the y-axis is $|\cos\theta|$ and the vertical distance from the x-axis is $|\sin\theta|$. These two distances form the legs of a right-angled triangle with hypotenuse 1 (the radius).

7.2 Compound Angle Formulas

These are used to combine or separate sine and cosine terms:

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B,$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B,$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad \left(A \pm B \neq \left(k + \frac{1}{2}\right)\pi\right).$$

7.3 Sum-to-Product Formulas

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2},$$
$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2},$$
$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2},$$
$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}.$$

8. Small-Angle Approximations

For very small angles measured in **radians**, we have powerful approximations:

$$\sin\theta \approx \theta, \qquad \tan\theta \approx \theta, \qquad \sec\theta \approx 1 + \frac{\theta^2}{2}.$$

9. Solving Any Triangle: Sine and Cosine Rules

Not all triangles in engineering problems are right-angled. For any triangle with sides a, b, c opposite angles A, B, C respectively:

The Sine Rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

The sine rule is used when we know two angles and any side (AAS or ASA), or when we know two sides and a non-included angle (SSA – the ambiguous case, which may give zero, one, or two possible triangles).

The Cosine Rule

$$a^2 = b^2 + c^2 - 2bc\cos A,$$

The cosine rule is used when we know two sides and the included angle (SAS), or when we know all three sides (SSS) – in which case it can be rearranged to find any angle.

Note, when $A = 90^\circ$, $\cos A = 0$, so the cosine rule reduces to $a^2 = b^2 + c^2$ – Pythagoras' theorem. The cosine rule is therefore a generalisation of Pythagoras for any triangle.