INSTITUTE OF
MATHEMATICS &
ITS APPLICATIONS

Trigonometry – Problems

1. In a surveying task for a future bridge project, a student engineer stands on one bank of a river and measures the angle of elevation to the top of a transmission tower on the opposite bank as 27°. She walks 95m directly away from the river in a straight line and now measures the angle of elevation as 18°. Assuming the ground is flat and the tower is vertical, what is the height h of the tower, to the nearest metre?

2. A radio telescope dish has a parabolic shape. The cross-section through its centre is modelled by:

$$y(x) = 0.25x^2, \quad -8 \leq x \leq 8$$

where x and y are in metres. The dish is supported by an external framework that connects to the dish at two points, P and Q, located at $x = -4$ and $x = 4$ respectively. In addition, an internal strut runs in a straight line from P to Q.

   (a) Sketch the function $y(x)$ in the domain stated and label the points P and Q with their coordinates.
   (b) Find the equation of the internal strut PQ.
   (c) The angle θ between the dish's tangent at P and the internal strut PQ determines the stress on the mounting. Find θ in degrees to one decimal place.
   (d) To reduce stress, engineers wish to adjust the external framework so that it is perpendicular to the dish at both attachment points. What would the gradient of the new external support need to be at P?

3. During a system overload, a machine component was violently ejected from its housing. Its subsequent parabolic trajectory, starting from the ejection point $(0, 0)$, is modelled by:

$$x(t) = 12\cos(15°)\,t, \qquad y(t) = 12\sin(15°)\,t - 4.9t^2$$

where x is the horizontal distance in metres, y is the height in metres, and t is the time in seconds.

   (a) Compare these equations to the standard SUVAT forms for projectile motion. State the component's initial speed and the ejection angle relative to the horizontal. Briefly explain why the $y(t)$ equation contains a quadratic term while the $x(t)$ equation does not.
   (b) Eliminate the parameter t from $x(t)$ and $y(t)$ to show that the trajectory of the component follows the path:

$$y = \frac{x}{\tan(15°)} - \frac{4.9x^2}{144\cos^2(15°)}$$

   (c) Using the exact trigonometric values:

$$\tan(15°) = 2 - \sqrt{3} \quad \text{and} \quad \cos^2(15°) = \frac{2 + \sqrt{3}}{4}$$

   rewrite the trajectory equation from part (b), without any trigonometric functions, such that all denominators are integers.
   (d) Using the simplified exact equation from part (c), determine the horizontal distance from the machine, measured in metres to two decimal places, at which the component strikes the ground.

4. An oceanographer stands on the deck of a research ship, with her eyes at a height h above sea level. The Earth is modelled as a sphere of radius R.

   (a) Draw a clear sketch showing:
      i. The centre of the Earth,

   ii. The observer at height h above the surface,
   iii. The point on the sea surface where the line of sight is tangent to the Earth (the horizon).
   Mark the right angle at the horizon point.

(b) Using Pythagoras' theorem on the right-angled triangle formed, derive an expression for the straight-line distance d from the observer to the horizon in terms of R and h.
(c) Given the Earth's radius $R = 6371\,\text{km}$, use graphing software to plot the horizon distance d (in km) as a function of height h (in m) for $0 \le h \le 100\,\text{m}$. State, looking from a height of $2\,\text{m}$ above sea level, how far away is the horizon (to 3 significant figures)?
(d) A coastal lighthouse needs to be visible 20 nautical miles ($\approx 37\,\text{km}$) out to sea. Calculate the minimum height of the lighthouse light, to the nearest metre above sea level, if atmospheric refraction extends the horizon by 8%.

5. In a lens coating facility, engineers analyse optical path differences to ensure interference effects meet design specifications. For a specific anti-reflective coating, the path difference Δ between two light rays is given by:

$$\Delta = d(\sec\theta - 1)$$

where $d = 0.1\,\mu\text{m}$ is the coating thickness, and θ is the angle of incidence in radians.

(a) Using the small-angle approximation $\sec\theta \approx 1 + \frac{\theta^2}{2}$, derive a simplified expression for the path difference Δ.
(b) For $\theta = 5^\circ$, calculate the percentage error between the exact and approximate values of Δ.

When light passes from one medium into another, it bends according to Snell's Law:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

(c) For $n_1 = 1.0$ (air), $n_2 = 1.5$ (glass), and $\theta_1 = 3^\circ$, find θ_2 using the small-angle approximation $\sin\theta \approx \theta$. Comment on the approximation's validity in this case.

The manufacturing division requires that for production-line alignment calculations, the small-angle approximation error for path difference must be less than 0.1% of the exact value.

(d) Write down the inequality that expresses this condition.
(e) Using numerical trial-and-error, or graphing software, determine the maximum allowable angle $\theta_{\max}$ (in degrees) that satisfies this 0.1% tolerance requirement.

6. A small sensor moves on a circular path of radius 1m in a horizontal plane, centred at the origin O. Its position at time t seconds is modelled by the point P with co-ordinates $(\cos\theta, \sin\theta)$, where θ is the angle (in radians) measured from the positive x-axis to the line OP. A receiver is fixed at the point A with co-ordinates $(2, 0)$.

(a) Verify, using a suitable trigonometric identity, that the point P traverses a circle of radius 1 centred on the origin.
(b) Show that the distance AP between the receiver and the sensor can be written as

$$AP^2 = 5 - 4\cos\theta.$$

(c) The signal strength S received at A is inversely proportional to the square of the distance from the sensor:

$$S = \frac{k}{AP^2}$$

for some constant k. The sensor is said to be in the "high-signal zone" when $S > \dfrac{k}{3}$.
By expressing S in terms of $\cos\theta$, find the range of values of θ (in radians, within $0 \le \theta < 2\pi$) for which the sensor is in the high-signal zone.
(d) Sketch the unit circle and highlight the arc(s) where the sensor is in the high-signal zone. Clearly label the principal angles on your diagram.

7. In a university laboratory, a single-phase AC voltage supply is described by

$$v(t) = 170\cos(100\pi t + 30^\circ)$$

where voltage is in volts and time t is in seconds.

(a) By plotting the function $v(t)$, state the peak (maximum) voltage and the phase angle of this supply.

The root-mean-square (RMS) voltage is the DC-equivalent voltage that produces the same heating effect in a resistor. It is given by

$$V_{\text{RMS}} = \frac{V_{\text{peak}}}{\sqrt{2}}.$$

In the remaining parts of this question, give your answers to one decimal place.

(b) Calculate the RMS voltage of this supply.
(c) On a phasor diagram, this source is represented as a vector of length 170 at an angle of 30° to the horizontal. Find the horizontal and vertical components, v_x and v_y, of this phasor.
(d) A second AC source has the same frequency, a peak voltage of 120 V, and a phase angle of −15°. Draw its phasor on the same diagram, and find its horizontal and vertical components.
(e) If the two sources are connected in series so that their voltages add as vectors, find the magnitude of the resultant peak voltage.
(f) Convert the resultant peak voltage found in part (e) to an RMS voltage using the formula given earlier.

8. In a car gearbox, meshing gear teeth generate harmonic forces. The net tangential force F on a gear at angular position ϕ is the superposition of two pressure components from the driving and driven gear teeth:

$$F(\phi) = 5\sin(\phi + 25^\circ) + 3\sin(\phi - 20^\circ)$$

(a) Using appropriate trigonometric identities, show that

$$F(\phi) = R\sin(\phi + \gamma)$$

stating expressions for the amplitude R and phase angle γ in terms of numerical values.
(b) Calculate the amplitude R to three significant figures, and the phase shift γ to one decimal place (in degrees).
(c) Without using a calculator, state the maximum and minimum values of $F(\phi)$, and the value(s) of ϕ at which the maximum occurs.
(d) The gearbox manufacturer adds a constant damping force of 1.2 units. Write down the new expression for the total force and state its maximum value.
(e) Sketch the graph of $F(\phi)$ for $0^\circ \le \phi \le 360^\circ$. On the same axes, sketch the graph after the damping force is added. On each sketch, clearly mark the amplitude, period, phase shift, and maximum and minimum points.