INSTITUTE OF MATHEMATICS & ITS APPLICATIONS

Integration – Tutor Notes

This session builds on the integration concepts introduced in the prereading, providing additional examples and explanations to reinforce understanding. Begin by assessing students' current knowledge – ask what they recall about integration from A-level or from the prereading. Use this to identify which sections may need clarification. If students express uncertainty about a particular topic, or indicate they found a section challenging, you can navigate directly to the relevant guidance below. The material is structured to follow the prereading sequence, allowing you to address specific gaps based on your student cohort's confidence levels.

Part I

Welcome Back – 15 Minutes

Integration as reverse differentiation: Establish that integration is the opposite of differentiation – it retrieves the original function $y = f(x)$ when given $f'(x) = \frac{dy}{dx}$. Begin by differentiating the following functions:

- $y = -2x^2 \quad \Rightarrow \quad \frac{dy}{dx} = -4x$
- $y = 5 - 2x^2 \quad \Rightarrow \quad \frac{dy}{dx} = -4x$
- $y = -2x^2 + c$ (where c is any real constant) $\quad \Rightarrow \quad \frac{dy}{dx} = -4x$

Point out that all three differentiate to the same result: $\frac{dy}{dx} = -4x$. Ask your students: "If I only told you that $\frac{dy}{dx} = -4x$, which of these is the original function?" They should realise it could be any of them – or indeed infinitely many possibilities. This is why integration must include a constant: $\int -4x\, dx = -2x^2 + C$. The constant accounts for the information lost during differentiation.

The constant of integration and initial conditions: Explain that while we can't know C from the derivative alone, we can determine it if we have one point on the original function – an initial condition. Open Desmos or GeoGebra and plot $y = -2x^2 + c$, adding a slider for c. Add a fixed point, say $(1, 1)$. Ask students: "What value of c makes the curve pass through this point?" Slide the slider until $c = 3$ (since $1 = -2(1)^2 + c \Rightarrow c = 3$). This visual demonstration shows how an initial condition picks one specific curve from an infinite family.

Basic rules of integration: Quickly review the power rule, constant multiple rule, and sum/difference rule from Section 3 of the prereading. Work through these examples with students, getting them to attempt each before revealing the answer:

a) $\int x\, dx = \frac{x^2}{2} + C$
b) $\int x^2\, dx = \frac{x^3}{3} + C$
c) $\int (x + 5)\, dx = \frac{x^2}{2} + 5x + C$
d) $\int 2x^3\, dx = 2 \cdot \frac{x^4}{4} + C = \frac{x^4}{2} + C$
e) $\int (-2x^3 + x^2)\, dx = -2 \cdot \frac{x^4}{4} + \frac{x^3}{3} + C = -\frac{x^4}{2} + \frac{x^3}{3} + C$
f) $\int 5\, dx = 5x + C$

Emphasise neat working – integration can get messy, so good habits early help.

Definite integrals and area: Introduce the notation $\int_a^b f(x)\,dx$, pointing out the two numbers (limits) at the top and bottom of the integral sign. Refer to the Fundamental Theorem of Calculus from the prereading:

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F(x)$ is any antiderivative of $f(x)$.
Ask the key question: "What happens to the $+C$?" Work through the logic: $F(b)+C-(F(a)+C) = F(b)-F(a)$ – the constants cancel. This is why definite integrals don't include $+C$.
To make this concrete, use the simplest example: $\frac{dy}{dx} = 5 \Rightarrow y = 5x + C$. Sketch $y = 5$ (a horizontal line). Introduce limits $x = 1$ and $x = 2$. Then:

$$\int_1^2 5\,dx = [5x]_1^2 = 5(2) - 5(1) = 5.$$

Shade the area under $y = 5$ between $x = 1$ and $x = 2$ – it's a rectangle of width 1 and height 5, area 5. This visually connects the definite integral to area under the curve.

Kinematics: Remind students that integration is the reverse of the differentiation journey they saw in the differentiation session. Draw attention to its relevance in A-level Mechanics:

$$a(t) \xrightarrow{\int} v(t) = \int a(t)\,dt + C_1 \xrightarrow{\int} s(t) = \int v(t)\,dt + C_2$$

The initial conditions come from $s(0)$ and $v(0)$. Emphasise two key interpretations:

- The area under an acceleration-time graph gives the change in velocity.
- The area under a velocity-time graph gives the displacement.

Trigonometric and exponential functions: Refer to Section 6 of the prereading to show:

$$\int \sin x\,dx = -\cos x + C, \quad \int \cos x\,dx = \sin x + C, \quad \int e^{kx}\,dx = \frac{1}{k}e^{kx} + C, \quad \int \frac{1}{x}\,dx = \ln|x| + C.$$

A useful exercise: ask students to differentiate the right-hand sides to verify they get back the integrands. For the exponential case, this clearly shows where the $\frac{1}{k}$ factor comes from – differentiating $\frac{1}{k}e^{kx}$ gives e^{kx}. This "check by differentiating" is a powerful habit to instil.

Advanced techniques – substitution and parts: Briefly introduce the ideas in Section 8 of the prereading. Work through the examples provided:

- **Substitution:** $\int \cos 2x\,dx$. Let $u = 2x$, $du = 2dx \Rightarrow dx = du/2$. Then $\int \cos u \cdot \frac{du}{2} = \frac{1}{2}\sin u + C = \frac{1}{2}\sin 2x + C$.
- **Integration by parts:** $\int x\cos x\,dx$. Choose $u = x$, $dv = \cos x\,dx$. Then $du = dx$, $v = \sin x$. So $\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$.

Reassure students that these techniques will be needed in later questions, but they don't need to master them immediately – they can refer back to the prereading. The key is recognising when each technique is appropriate.

Part II

Getting Started – 10 Minutes

Let students work on **Question 1**, then walk through the answers. Pay particular attention to the abstract nature of the question – it contains only one number (20 m), which students may find unsettling. Reassure them that this is typical of modelling; we introduce the necessary numbers through assumptions. Prompt them to identify these assumptions: constant acceleration $g = 9.8$ m/s^2, no air resistance, and release from rest ($v(0) = 0$). Also, check they're explicitly finding C_1 and C_2 – it's easy to skip the $+C$ here since both are zero, but this misses a key point. Ensure they understand why $s(t) = 20$: displacement is measured from the release point, and hitting ground means travelling 20 m downward. If students jump straight to SUVAT equations, redirect them to use integration, while acknowledging the method but reminding them integration is more powerful and essential when acceleration isn't constant, which appears in later questions.

Getting Stuck In – 30 Minutes

Ask students to focus on **Questions 2 to 4**. Pay special attention to:

- **Q2:** Students may struggle with the concept of integrating a moment to get slope – emphasise that $\frac{d\theta}{dx} = M(x)/EI$ is given, so $\theta(x)$ is simply the integral. Watch for forgetting the constant of integration; remind them that $\theta(0) = 0$ determines it. The result $\theta(4) = 1/150$ rad is very small – this surprises students and leads nicely into parts (d) and (e). For part (f), prompt them to think about real design trade-offs: stiffness vs cost, weight, aesthetics.
- **Q3:** Since the maximum is at the endpoint $t = 20$, this is a good moment to discuss endpoint maxima. Watch for algebra errors when integrating the fraction $\frac{20-t}{10}$. In part (d), some may forget that fuel consumption rate is given and simply need integrating. Emphasise that what's given is the *rate* at which fuel is consumed, hence the need to integrate over the time interval. Part (e)'s fuel efficiency (kg/m) is a meaningful performance metric – ask what it represents (fuel used per metre climbed).
- **Q4:** The symmetry argument in part (a) is key – draw the sine wave and shade areas to show they cancel. In part (b), students often forget the $-\frac{1}{k}$ factor when integrating $\sin(kt)$. Part (d) tests understanding of frequency change – doubling frequency halves the period, so the new function is $0.8\sin(2\pi t/3)$. The resulting volume halves, which makes physiological sense.

Break

Encourage students to step away from screens briefly.

<u>Part III</u>

More Problem Solving – 30 Minutes

Ask students to focus on **Questions 5 to 7** in that order.

- **Q5:** Students may treat the acceleration and deceleration phases separately but forget to connect them – the velocity at $t = 20$ becomes the initial velocity for deceleration. Watch for sign errors in part (d); deceleration should be negative. Point out that part (d) involves constant/uniform deceleration, so SUVAT equations apply here – unlike the initial phase where acceleration was given as a function of t and required integration.
- **Q6:** Differentiating $V(t) = V_0(1 - e^{-t/RC})$ requires the chain rule; show this explicitly. The time constant $\tau = RC = 20$ s is a key concept – at $t = \tau$, current has dropped to $1/e \approx 37\%$ of its initial value. Part (d)'s $\frac{dI}{dt}$ has units A/s – ask what this represents (rate of change of current).
- **Q7:** This may be the most challenging question in the session. Students may not understand what creep is – be prepared to explain it in simple terms ("time-dependent deformation under constant stress" may be inaccessible). Both parts (a) and (c) require integrating exponential functions, and part (c) may seem particularly abstract with two unknowns – expect students to struggle here. Part (d) is algebraically demanding – letting $x = e^{-10k}$ simplifies the simultaneous equations. Take a moment to share this substitution technique; it makes the working much easier to follow.

Wrap Up – 5 Minutes

Pose any remaining questions as extensions:

- **Q8:** Part (a) tests whether students recognise that the sine term integrates to zero over a full period, this is a key insight. Part (e) is the most technically demanding, requiring integration by parts; signpost students to the prereading material for guidance on this technique.

Remind students that they can attempt these in their own time. Encourage them to use the solutions only after attempting the problems themselves.