INSTITUTE OF MATHEMATICS & ITS APPLICATIONS

Integration – Solutions

1. In an abuse test case, a sensor is dropped from a height of 20 m to see if its casing cracks on impact. Using integration, calculate the time taken for the sensor to hit the ground, clearly stating any physical assumptions made in your model.

Solution:

Acceleration is constant: $a = g = 9.8$ ms^{-2}. Integrate to obtain velocity:

$$v(t) = \int a\,dt = \int g\,dt = gt + C_1.$$

At $t = 0$, $v(0) = 0 \Rightarrow C_1 = 0$. Hence $v(t) = gt$.
Integrate again to obtain displacement:

$$s(t) = \int v(t)\,dt = \int gt\,dt = \frac{1}{2}gt^2 + C_2.$$

At $t = 0$, the sensor is at the top, so we set $s(0) = 0$ (taking the release point as origin).
Thus $C_2 = 0$, giving

$$s(t) = \frac{1}{2}gt^2.$$

The ground is at $s = 20$ m. Set $s(t) = 20$:

$$\frac{1}{2}gt^2 = 20 \quad \Longrightarrow \quad t^2 = \frac{40}{g}.$$

Substitute $g = 9.8$:

$$t^2 = \frac{40}{9.8} \approx 4.0816 \quad \Longrightarrow \quad t \approx \sqrt{4.0816} \approx 2.02 \text{ s}.$$

$$\boxed{t \approx 2.02 \text{ s}}.$$

Note: The integration method used here is equivalent to the standard SUVAT equation $s = ut + \frac{1}{2}at^2$ with $u = 0$. The assumptions of no air resistance and constant g are idealisations; in reality, air resistance might slightly increase the time.

2. You are an engineer designing an iron-work balcony for a café. The balcony projects 4 m from the wall and supports flower boxes along its entire length. The flower boxes create a bending moment in the beam of the form $M(x)$, where x is the distance in metres from the wall.
The tilt (slope) of the balcony beam, $\theta(x)$, is related to the bending moment by

$$\frac{d\theta}{dx} = \frac{M(x)}{EI},$$

where $EI = 2400\,\text{kNm}^2$ is the beam's flexural rigidity (a measure of its resistance to bending). The bending moment (kNm) in the beam is given by:

$$M(x) = 6x - 1.5x^2, \quad 0 \le x \le 4.$$

(a) Write down an expression for $\frac{d\theta}{dx}$ in terms of $M(x)$ and EI.
(b) Given that the beam is built perfectly horizontal at the wall, so $\theta(0) = 0$, determine $\theta(x)$.

(c) Find the tilt at the free end of the balcony, $x = 4m$. Give your answer in both radians and degrees.

The building regulations state that the maximum allowable slope for a public balcony is $2^\circ \approx 0.035$ rad. The café owner believes that slopes greater than $1.5^\circ \approx 0.026$ rad are visibly noticeable and would make the flower boxes look crooked.

(d) Using your result from part (c), decide whether the balcony satisfies the safety regulation, and whether the owner would notice the tilt just by looking at the flower boxes.

(e) Using a graphing software to plot $\theta(x)$, justify why checking only the tilt at the free end $x = 4m$ is sufficient to answer the safety and aesthetic questions in part (d).

(f) Prove that a higher EI (a stiffer beam) makes the tilts smaller. Briefly explain why, in practice, a designer might still choose a lower EI (a more flexible beam) even though it leads to larger tilts.

Solution:

(a) Expression for $\frac{d\theta}{dx}$

$$\frac{d\theta}{dx} = \frac{M(x)}{EI} = \frac{6x - 1.5x^2}{2400}.$$

(b) Determine $\theta(x)$ given $\theta(0) = 0$

Integrate with respect to x:

$$\theta(x) = \int \frac{6x - 1.5x^2}{2400}\, dx = \frac{1}{2400}\left(3x^2 - 0.5x^3\right) + C.$$

Using $\theta(0) = 0$ gives $C = 0$. Hence

$$\theta(x) = \frac{3x^2 - 0.5x^3}{2400}.$$

(c) Tilt at the free end $x = 4$ m

Substitute $x = 4$:

$$\theta(4) = \frac{3(4)^2 - 0.5(4)^3}{2400} = \frac{1}{150} \text{ rad}.$$

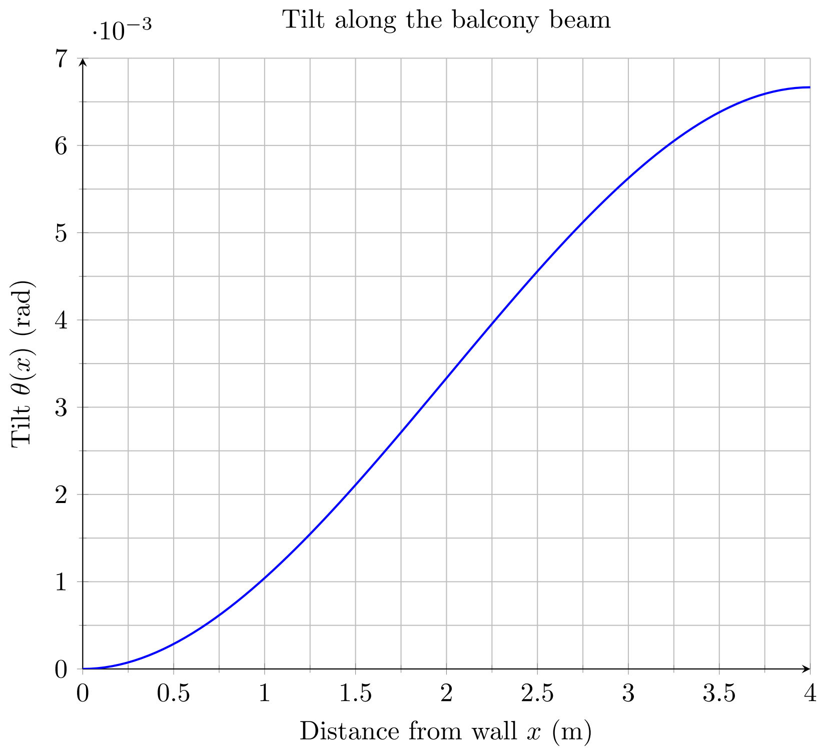
In degrees: $\theta(4) \times \frac{180}{\pi} \approx 0.006667 \times 57.2958 \approx 0.382^\circ$.

$$\boxed{\theta(4) = \frac{1}{150} \text{ rad} \approx 0.382^\circ}.$$

(d) Safety regulation and owner's notice

The calculated tilt of 0.00667 rad is well below the safety limit of $2^\circ \approx 0.035$ rad. Similarly, it is far smaller than the owner's notice threshold of $1.5^\circ \approx 0.026$ rad, meaning the tilt would not be visible to someone looking at the flower boxes.

(e) Graph of $\theta(x)$ and justification



The graph shows that $\theta(x)$ is increasing on $[0, 4]$ (the derivative $\frac{d\theta}{dx} = M/EI$ is positive because $M(x) > 0$ for $0 < x < 4$). Therefore the maximum tilt occurs at the free end $x = 4$, when $\frac{d\theta}{dx} = 0$. Checking only the tilt at the free end is sufficient because it is the largest tilt anywhere on the beam; if it satisfies the regulations and is below the notice threshold, then all points along the beam also satisfy them.

(f) Effect of higher EI and design choice
From $\theta(x) = \frac{1}{EI} \int M(x)\,dx$, a higher EI (stiffer beam) makes $\theta(x)$ smaller for the same moment.
Why might a designer choose a lower EI? A more flexible beam (lower EI) may be:

- Lighter and cheaper, reducing material and support costs.
- Easier to fabricate or install.
- Required for aesthetic reasons (e.g., a slender appearance).

However, larger tilts must still remain within safety limits and acceptable deflection standards. The designer balances stiffness against cost, weight, and architectural requirements.

3. An aircraft's vertical acceleration, measured in ms^{-2}, during its climb to cruise altitude is modelled by:

$$a(t) = \frac{20 - t}{10}, \quad 0 \le t \le T.$$

Given that the initial vertical velocity $v(0) = 8\ \text{ms}^{-1}$ and the initial altitude $h(0) = 200\,\text{m}$:

(a) Find the time T when the climb phase ends.

(b) By determining the velocity function $v(t)$ for $0 \le t \le T$, determine the maximum vertical velocity reached.

(c) Calculate the cruising altitude at time T.

The rate of fuel consumption, $F(t)$ $\left(\text{kgs}^{-1}\right)$ at time t, during climb is proportional to the vertical acceleration:

$$F(t) = 2 + 0.5a(t).$$

(d) Find the total fuel burnt during the climb phase.

(e) Fuel efficiency, measured as the amount of fuel required to gain one meter of altitude, is a critical performance metric for aircraft during the climb phase. Determine the fuel efficiency.

Solution:

(a) Time T when climb phase ends
The climb phase ends when acceleration becomes zero (cruise condition):

$$a(t) = 0 \implies \frac{20-t}{10} = 0 \implies t = 20 \text{ s}.$$

$$\boxed{T = 20 \text{ s}}.$$

(b) Velocity function and maximum velocity
Integrate acceleration to find velocity:

$$v(t) = \int a(t)\,dt = \int \frac{20-t}{10}\,dt = 2t - \frac{t^2}{20} + C.$$

Using $v(0) = 8$ gives $C = 8$. Hence

$$v(t) = 8 + 2t - \frac{t^2}{20} \text{ ms}^{-1}.$$

This is a quadratic opening downward. The maximum occurs where $\frac{dv}{dt} = a(t) = 0$, i.e. at $t = 20$ (the endpoint). Therefore

$$v_{\max} = v(20) = 8 + 2(20) - \frac{20^2}{20} = 28 \text{ ms}^{-1}.$$

$$\boxed{v_{\max} = 28 \text{ ms}^{-1}}.$$

(c) Cruising altitude at time T

Integrate velocity to obtain altitude:

$$h(t) = \int v(t)\,dt = \int \left(8 + 2t - \frac{t^2}{20}\right) dt = 8t + t^2 - \frac{t^3}{60} + D.$$

With $h(0) = 200$, we have $D = 200$. Thus

$$h(t) = 200 + 8t + t^2 - \frac{t^3}{60}.$$

At $t = 20$:

$$h(20) = 200 + 8(20) + 20^2 - \frac{20^3}{60} = \frac{1880}{3} \text{ m}.$$

Numerically, $h(20) \approx 626.67$ m.

$$\boxed{h(20) = \frac{1880}{3} \text{ m} \approx 626.7 \text{ m}}.$$

(d) Total fuel burnt during climb

Fuel consumption rate: $F(t) = 2 + 0.5a(t)$. Substituting $a(t)$:

$$F(t) = 2 + 0.5\left[\frac{20-t}{10}\right] = 3 - \frac{t}{20} \text{ kgs}^{-1}.$$

Total fuel burnt from $t = 0$ to $t = 20$:

$$\int_0^{20} F(t)\,dt = \int_0^{20} \left(3 - \frac{t}{20}\right) dt = \left[3t - \frac{t^2}{40}\right]_0^{20} = 3(20) - \frac{400}{40} = 50 \text{ kg}.$$

$$\boxed{\text{Fuel burnt} = 50 \text{ kg}}.$$

(e) Fuel efficiency (fuel per metre of altitude gained)
Altitude gained during climb:

$$\Delta h = h(20) - h(0) = \frac{1880}{3} - 200 = \frac{1280}{3} \text{ m.}$$

Fuel efficiency η is fuel used per metre gained:

$$\eta = \frac{50}{\Delta h} = \frac{50}{\frac{1280}{3}} = \frac{15}{128} \text{ kgm}^{-1}.$$

$$\boxed{\eta = \frac{15}{128} \text{ kg/m} \approx 0.117 \text{ kgm}^{-1}}.$$

4. A patient on a ventilator has their airflow rate (in litres per second) modelled by:

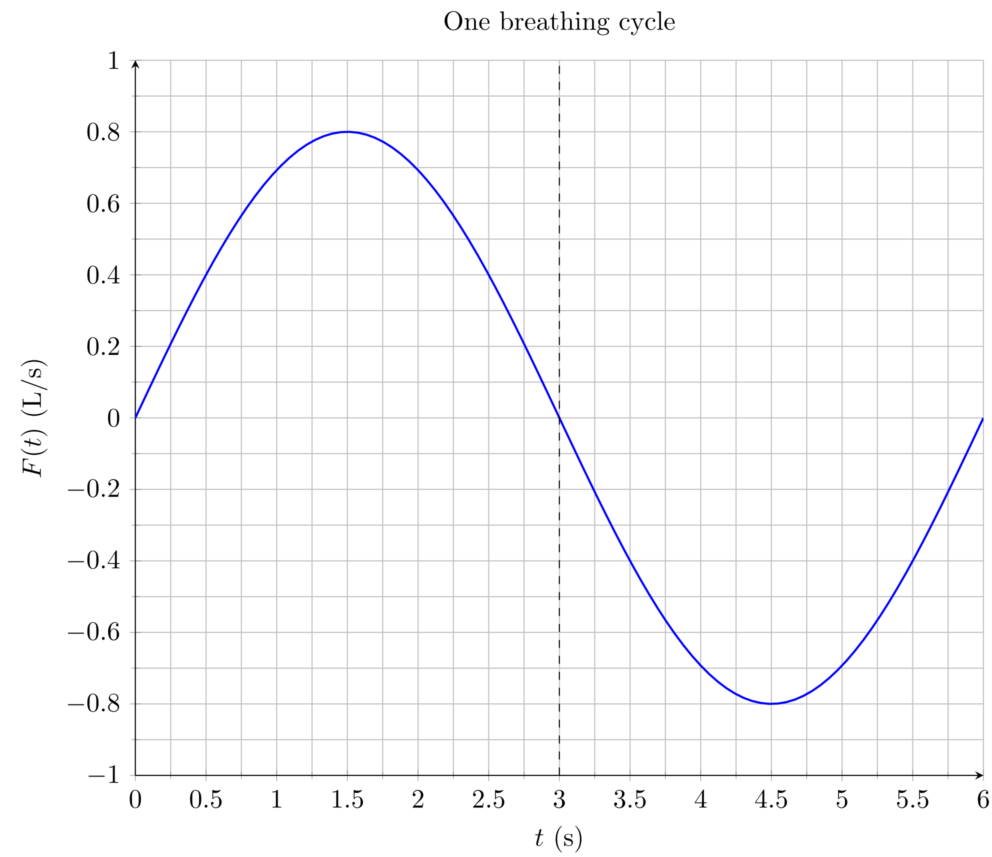
$$F(t) = 0.8 \sin\left(\frac{\pi t}{3}\right) \quad 0 \leq t \leq 6,$$

where t is time in seconds.

(a) By sketching the function $F(t)$ over one breathing cycle, explain why the net volume over the full cycle is zero.
(b) Find the total volume of air inhaled during one 6 second cycle.
(c) Determine the average flow rate over the inhalation phase ($0 \leq t \leq 3$).
(d) The ventilator adjusts to double the frequency. Express the new flow rate function for a 3 second cycle and find the new total volume per cycle.

Solution:

(a) Sketch and net volume



Over one full cycle $0 \leq t \leq 6$, the flow is positive during inhalation $(0 < t < 3)$ and negative during exhalation $(3 < t < 6)$. The net volume change is the integral of flow, which is zero because the areas above and below the time axis are equal due to symmetry of the sine wave. Hence the net volume over the full cycle is zero.

(b) Total volume inhaled during one cycle
Inhalation occurs for $0 \le t \le 3$. The inhaled volume is

$$V_{\text{inh}} = \int_0^3 F(t)\,dt = \int_0^3 0.8\sin\left(\frac{\pi t}{3}\right) dt.$$

Let $k = \frac{\pi}{3}$. Then

$$\int \sin(kt)\,dt = -\frac{1}{k}\cos(kt).$$

Hence

$$V_{\text{inh}} = 0.8\left[-\frac{1}{k}\cos(kt)\right]_0^3 = -\frac{0.8}{k}\Big[\cos(kt)\Big]_0^3.$$

Substitute $k = \pi/3$:

$$V_{\text{inh}} = -\frac{0.8}{\pi/3}\left[\cos\left(\frac{\pi t}{3}\right)\right]_0^3 = -\frac{2.4}{\pi}\left[\cos\left(\frac{\pi t}{3}\right)\right]_0^3.$$

Evaluate the cosine at the limits:

$$\cos\left(\frac{\pi \cdot 3}{3}\right) = \cos\pi = -1, \qquad \cos(0) = 1.$$

Therefore

$$V_{\text{inh}} = -\frac{2.4}{\pi}\big[(-1) - (1)\big] = -\frac{2.4}{\pi}(-2) = \frac{4.8}{\pi}\ \text{L}.$$

Numerically, $V_{\text{inh}} \approx 1.53$ L.

$$\boxed{V_{\text{inh}} = \frac{4.8}{\pi}\ \text{L}}.$$

(c) Average flow rate during inhalation

The inhalation phase lasts 3 seconds, so the average flow rate is

$$\bar{F} = \frac{1}{3}\int_0^3 F(t)\,dt = \frac{1}{3} \times V_{\text{inh}} = \frac{1}{3} \times \frac{4.8}{\pi} = \frac{1.6}{\pi}\ \text{Ls}^{-1}.$$

$$\boxed{\bar{F} = \frac{1.6}{\pi}\ \text{Ls}^{-1} \approx 0.509\ \text{Ls}^{-1}}.$$

(d) Doubling the frequency

The original period is $T = 6$s. Doubling the frequency gives a new period $T_{\text{new}} = 3$s. The angular frequency becomes $\omega_{\text{new}} = 2\pi/T_{\text{new}} = 2\pi/3$. The new flow rate function for one 3-second cycle is

$$F_{\text{new}}(t) = 0.8\sin\left(\frac{2\pi t}{3}\right), \qquad 0 \le t \le 3.$$

Inhalation now occurs during $0 \le t \le 1.5$s. Following the same method as in part (b):

$$V_{\text{inh,new}} = \int_0^{1.5} 0.8\sin\left(\frac{2\pi t}{3}\right) dt.$$

With $k = 2\pi/3$, the integral is $-\frac{1}{k}\cos(kt)$. Evaluating:

$$V_{\text{inh,new}} = -\frac{0.8}{2\pi/3}\left[\cos\left(\frac{2\pi t}{3}\right)\right]_0^{1.5} = -\frac{1.2}{\pi}\big[\cos\pi - \cos 0\big] = -\frac{1.2}{\pi}(-1-1) = \frac{2.4}{\pi}\ \text{L}.$$

$$\boxed{F_{\text{new}}(t) = 0.8\sin\left(\frac{2\pi t}{3}\right), \quad V_{\text{inh,new}} = \frac{2.4}{\pi}\ \text{L}}.$$

5. In a high storey building, an express elevator accelerates from rest with:

$$a(t) = 1 - 0.05t \quad 0 \le t \le 20,$$

where t is in seconds and the acceleration a has units ms^{-2}. The acceleration phase ends at $t = 20$s. Each floor is 4.0m high.

(a) Given that the initial velocity $v(0) = 0$ and the initial displacement $s(0) = 0$, find the velocity function $v(t)$ and position function $s(t)$.
(b) Calculate the velocity and height at $t = 20$s.
(c) How many floors has the elevator passed by $t = 20$s?

The elevator must stop exactly at the 40^{th} floor (160m above ground). After $t = 20$s, the elevator decelerates uniformly.

(d) Find the constant deceleration needed to stop exactly at 160m.
(e) How long does the deceleration phase last?
(f) What is the total journey time from the ground to the 40^{th} floor?

Solution:

(a) Velocity and position functions
Integrate acceleration to find velocity:

$$v(t) = \int a(t)\,dt = \int (1 - 0.05t)\,dt = t - 0.025t^2 + C.$$

Using $v(0) = 0$ gives $C = 0$. Hence

$$v(t) = t - 0.025t^2 \text{ ms}^{-1}.$$

Integrate velocity to find position:

$$s(t) = \int v(t)\,dt = \int (t - 0.025t^2)\,dt = \frac{t^2}{2} - \frac{0.025}{3}t^3 + D.$$

With $s(0) = 0$, we have $D = 0$. Thus

$$s(t) = \frac{t^2}{2} - \frac{0.025}{3}t^3 = \frac{t^2}{2} - \frac{t^3}{120} \text{ m}.$$

(b) Velocity and height at $t = 20$s

$$v(20) = 20 - 0.025(20)^2 = 10 \text{ ms}^{-1}.$$

$$s(20) = \frac{20^2}{2} - \frac{20^3}{120} = \frac{400}{3} \text{ m}.$$

$$\boxed{v(20) = 10 \text{ ms}^{-1}, \quad s(20) = \frac{400}{3} \text{ m} \approx 133.3 \text{ m}}.$$

(c) Number of floors passed by $t = 20$s

Each floor is 4m, so number of floors $= \frac{s(20)}{4} = \frac{400/3}{4} \approx 33.33$. Since it's not an integer, the elevator has passed 33 full floors and is partway through the 34th.

$$\boxed{\text{33 full floors}}.$$

(d) Constant deceleration needed to stop at the 40th floor

The 40th floor is at height $40 \times 4 = 160$m. After $t = 20$, the elevator must travel an additional distance

$$\Delta s = 160 - s(20) = 160 - \frac{400}{3} = \frac{80}{3} \text{ m}.$$

Initial velocity for this phase is $v_0 = 10\text{m/s}$, final velocity $v = 0$, constant deceleration a_d (negative). Using $v^2 = v_0^2 + 2a_d\,\Delta s$:

$$0 = 10^2 + 2a_d \cdot \frac{80}{3} \implies 2a_d \cdot \frac{80}{3} = -100 \implies a_d = -\frac{15}{8}\ \text{ms}^{-2}.$$

$$\boxed{a_d = -\frac{15}{8}\ \text{ms}^{-2}\ (= -1.875\ \text{ms}^{-2})}.$$

(e) Duration of deceleration phase

Using $v = v_0 + a_d t_d$:

$$0 = 10 - \frac{15}{8}t_d \implies t_d = \frac{16}{3}\ \text{s} \approx 5.33\ \text{s}.$$

$$\boxed{t_d = \frac{16}{3}\ \text{s}}.$$

(f) Total journey time

Total time $= 20 + t_d = 20 + \frac{16}{3} = \frac{76}{3}$ s ≈ 25.33 s.

$$\boxed{T_{\text{total}} = \frac{76}{3}\ \text{s}}.$$

6. An electrical circuit contains a capacitor which is being charged. At time t seconds, suppose that $Q(t)$ is the charge on the capacitor and the current $I(t)$, in amps, flowing into the capacitor is given by:

$$I(t) = 3e^{-\frac{t}{5}}, \quad t \geq 0.$$

Given $I = Q'(t)$ and the initial charge on the capacitor is zero:

(a) Find the charge $Q(t)$ on the capacitor at time t.
(b) Determine the total charge that accumulates on the capacitor over the first 10 seconds.
(c) Determine the limiting charge as $t \to \infty$.
(d) Suppose the capacitor is considered "fully charged" when it reaches 95% of its limiting charge. Find the time (to 2 decimal places) when the capacitor becomes fully charged.

Solution:

(a) Charge $Q(t)$ at time t

Since $Q'(t) = I(t)$, we integrate:

$$Q(t) = \int I(t)\,dt = \int 3e^{-t/5}\,dt.$$

Let $k = -\frac{1}{5}$, so $\int e^{kt}\,dt = \frac{1}{k}e^{kt}$. Thus

$$\int 3e^{-t/5}\,dt = 3 \cdot \frac{1}{-1/5}e^{-t/5} + C = -15e^{-t/5} + C.$$

Using $Q(0) = 0$:

$$0 = -15e^0 + C \implies C = 15.$$

Hence

$$Q(t) = 15\left(1 - e^{-t/5}\right)\ \text{C}.$$

(b) Total charge over first 10 seconds

$$Q(10) = 15\left(1 - e^{-10/5}\right) = 15\left(1 - e^{-2}\right)\ \text{C}.$$

$$Q(10) \approx 15(0.8647) = 12.97\ \text{C}.$$

(c) Limiting charge as $t \to \infty$
As $t \to \infty$, $e^{-t/5} \to 0$. Hence

$$\lim_{t\to\infty} Q(t) = 15 \text{ C}.$$

(d) Time to reach 95% of limiting charge

95% of limiting charge is $0.95 \times 15 = 14.25$ C. Set $Q(t) = 14.25$:

$$15\left(1 - e^{-t/5}\right) = 14.25 \implies 1 - e^{-t/5} = \frac{14.25}{15} = 0.95.$$

Thus $e^{-t/5} = 0.05$. Take natural logs:

$$-\frac{t}{5} = \ln(0.05) \implies t = -5\ln(0.05) = 14.9785 \text{ s}.$$

Rounded to two decimal places:

$$\boxed{t \approx 14.98 \text{ s}}.$$

7. Creep is the time-dependent deformation of materials under constant stress. For a certain nickel-based superalloy used in turbine blades, suppose ϵ denotes the strain and the creep strain rate (strain per hour) is given by:

$$\frac{d\epsilon}{dt} = 0.002t^{-0.5} \quad t \geq 0,$$

where t is time in hours. At $t = 1$ hour, the strain is measured as $\epsilon = 0.001$.

(a) Find the strain $\epsilon(t)$ for $t \geq 1$.
(b) Calculate the strain and strain rate after 100 hours of operation. Express your answers as percentages.

In a laboratory test on a different material, the creep strain rate decreases exponentially with time:

$$\frac{d\epsilon}{dt} = Be^{-kt}, \quad t \geq 0$$

where B and k are positive constants. The initial strain at $t = 0$ is zero.

(c) Find an expression for $\epsilon(t)$.
(d) Experimental measurements show that when $t = 10$ hours, $\epsilon = 0.005$, and when $t = 20$ hours, $\epsilon = 0.008$. Determine the values of B and k (to 3 significant figures).

An engineer is designing a pressure vessel that will operate at high temperature for 10,000 hours. The design specification requires that the total creep strain must not exceed 0.01.

(e) Using the power-law model from part (a), determine whether the strain limit is exceeded within the 10,000-hour service life.
(f) If the limit is exceeded, calculate at what time (to the nearest hour) the strain first reaches 0.01.

Solution:

(a) Find $\epsilon(t)$ for $t \geq 1$
Integrate the rate equation:

$$\epsilon(t) = \int 0.002\,t^{-0.5}\,dt = 0.002(2\,t^{0.5}) + C = 0.004\sqrt{t} + C.$$

Using $\epsilon(1) = 0.001$:

$$0.004(1) + C = 0.001 \implies C = -0.003.$$

Hence
$$\boxed{\epsilon(t) = 0.004\sqrt{t} - 0.003 \quad (t \geq 1)}.$$

(b) Strain and strain rate after 100 hours

Strain at $t = 100$:
$$\epsilon(100) = 0.004\sqrt{100} - 0.003 = 0.037.$$

As a percentage:
$$0.037 \times 100\% = \boxed{3.7\%}.$$

Strain rate at $t = 100$:
$$\left.\frac{d\epsilon}{dt}\right|_{t=100} = 0.002\,(100)^{-0.5} = 0.002 \times \frac{1}{10} = 0.0002\ \mathrm{h}^{-1}.$$

Expressed as a percentage, this is $0.0002 \times 100 = 0.02\%$ per hour.

$$\boxed{0.02\%\ \mathrm{h}^{-1}}.$$

(c) Expression for $\epsilon(t)$

Integrate:
$$\epsilon(t) = \int_0^t Be^{-k\tau}\,d\tau = B\left[-\frac{1}{k}e^{-k\tau}\right]_0^t = \frac{B}{k}\left(1 - e^{-kt}\right).$$
$$\boxed{\epsilon(t) = \frac{B}{k}\left(1 - e^{-kt}\right)}.$$

(d) Determine B and k given $\epsilon(10) = 0.005$ and $\epsilon(20) = 0.008$

Let $x = e^{-10k}$. Then $e^{-20k} = x^2$. From the formula:
$$\frac{B}{k}(1 - x) = 0.005, \qquad \frac{B}{k}(1 - x^2) = 0.008.$$

Dividing the second by the first:
$$\frac{1 - x^2}{1 - x} = \frac{0.008}{0.005} = 1.6.$$

Since $1 - x^2 = (1 - x)(1 + x)$, this simplifies to $1 + x = 1.6$, hence $x = 0.6$. Thus $e^{-10k} = 0.6 \Rightarrow -10k = \ln 0.6 \Rightarrow k = -\frac{\ln 0.6}{10} = \frac{\ln(5/3)}{10}$. Numerically, $\ln(5/3) \approx 0.510826$, so

$$k \approx 0.05108\ \mathrm{h}^{-1} \quad \text{(to 3 s.f. 0.0511)}.$$

Now from $\frac{B}{k}(1 - 0.6) = 0.005$:

$$\frac{B}{k} = 0.0125 \implies B = 0.0125k \approx 0.0125 \times 0.05108 = 0.0006385.$$

To three significant figures:

$$\boxed{B = 6.39 \times 10^{-4}\ \mathrm{h}^{-1}, \quad k = 0.0511\ \mathrm{h}^{-1}}.$$

(e) Check if strain limit is exceeded at $t = 10\,000$h

$$\epsilon(10000) = 0.004\sqrt{10000} - 0.003 = 0.397.$$

This is far above 0.01; therefore the limit is **exceeded**.

$$\boxed{\text{Yes, strain exceeds 0.01 (reaches 0.397)}}.$$

(f) Time when strain first reaches 0.01 (to nearest hour)

Set $\epsilon(t) = 0.01$:

$$0.004\sqrt{t} - 0.003 = 0.01 \implies \sqrt{t} = \frac{0.013}{0.004} = 3.25.$$

Thus $t = (3.25)^2 = 10.5625$h. To the nearest hour,

$$\boxed{t \approx 11 \text{ hours}}.$$

8. A factory has two processes that discharge wastewater into a treatment tank. The discharge rate $r_A(t)$, in litres per hour, from Process A is modelled by:

$$r_A(t) = 40 + 10\sin\left(\frac{\pi t}{12}\right),$$

where t is the time in hours since midnight. Process B discharges at a rate $r_B(t)$, in litres per hour, given by:

$$r_B(t) = 50e^{-0.1t}, \quad t \geq 0.$$

The treatment tank has a capacity of 2000 litres and is initially empty at $t = 0$.

(a) Calculate the total volume of wastewater discharged from Process A over the first 24 hours. Then, determine the average discharge rate of Process A over this period.
(b) Find the total volume discharged from process B from $t = 0$ to $t = 24$ hours.
(c) At what time (to 2 decimal places) does the total volume discharged from process B first reach 400 litres?
(d) Suppose both processes run simultaneously from $t = 0$. Write down an expression for the combined discharge rate $R(t)$ and determine whether the tank will overflow within the first 24 hours.

In an effort to reduce discharge, the factory installs a filter on Process A. The filter reduces the discharge rate by a factor of $\left(1 - \frac{t}{24}\right)$ for $0 \leq t \leq 24$.

(e) By expressing the modified discharge rate for Process A, calculate the total volume discharged from the modified Process A over 24 hours.

Solution:

(a) Total volume from Process A over first 24h and average rate

$$V_A(24) = \int_0^{24} r_A(t)\,dt = \int_0^{24} \left(40 + 10\sin\frac{\pi t}{12}\right) dt.$$

Compute each term separately:

$$\int_0^{24} 40\,dt = 40\,[t]_0^{24} = 40 \times 24 = 960.$$

For the sine term, use $\int \sin(kt)\,dt = -\frac{1}{k}\cos(kt)$ with $k = \frac{\pi}{12}$:

$$\int_0^{24} 10\sin\frac{\pi t}{12}\,dt = 10\left[-\frac{12}{\pi}\cos\frac{\pi t}{12}\right]_0^{24} = -\frac{120}{\pi}\left[\cos\left(\frac{\pi \cdot 24}{12}\right) - \cos(0)\right] = -\frac{120}{\pi}\left[\cos(2\pi) - \cos(0)\right] = 0.$$

The sine term integrates over a full period (24h) to zero, thus $V_A(24) = 960$L.
Average discharge rate:

$$\bar{r}_A = \frac{V_A(24)}{24} = \frac{960}{24} = 40 \text{ Lh}^{-1}.$$

$$\boxed{V_A(24) = 960 \text{ L}, \qquad \bar{r}_A = 40 \text{ Lh}^{-1}}.$$

(b) Total volume from Process B from $t = 0$ to $t = 24$h

$$V_B(24) = \int_0^{24} 50e^{-0.1t}\,dt = 50\left[-\frac{1}{0.1}e^{-0.1t}\right]_0^{24} = 500\left(1 - e^{-2.4}\right)\text{ L}.$$

Numerically, $e^{-2.4} \approx 0.0907$, so

$$V_B(24) \approx 500 \times 0.9093 = 454.65\text{ L}.$$

$$\boxed{V_B(24) \approx 454.7\text{ L}}.$$

(c) Time when Process B first reaches 400L

Let T satisfy $\int_0^T 50e^{-0.1t}\,dt = 400$:

$$500\left(1 - e^{-0.1T}\right) = 400 \implies 1 - e^{-0.1T} = 0.8 \implies e^{-0.1T} = 0.2.$$

Taking natural logs:

$$-0.1T = \ln 0.2 \implies T = -10\ln 0.2 \approx 16.0944$$

$$\boxed{T \approx 16.09\text{ h}}.$$

(d) Combined discharge and tank overflow within 24h

Combined rate: $R(t) = r_A(t) + r_B(t)$. Total volume after 24h:

$$V_{\text{total}}(24) = V_A(24) + V_B(24) = 960 + 500\left(1 - e^{-2.4}\right) \approx 1414.7\text{ L}.$$

This is less than the tank capacity 2000L. Since the volume increases monotonically, it never exceeds capacity. Hence the tank does **not** overflow.

$$\boxed{\text{No overflow; final volume } \approx 1415\text{ L} < 2000\text{ L}}.$$

(e) Modified Process A with filter

The filter reduces the discharge rate by factor $\left(1 - \frac{t}{24}\right)$ for $0 \le t \le 24$. Thus

$$r_A^{\text{mod}}(t) = \left(40 + 10\sin\frac{\pi t}{12}\right)\left(1 - \frac{t}{24}\right).$$

Total volume over 24h:

$$V_A^{\text{mod}} = \int_0^{24}\left(40 + 10\sin\frac{\pi t}{12}\right)\left(1 - \frac{t}{24}\right)dt.$$

Split the integral:

$$V_A^{\text{mod}} = \int_0^{24} 40\left(1 - \frac{t}{24}\right)dt + \int_0^{24} 10\sin\frac{\pi t}{12}\left(1 - \frac{t}{24}\right)dt.$$

First term:

$$\int_0^{24} 40\left(1 - \frac{t}{24}\right)dt = 40\left[t - \frac{t^2}{48}\right]_0^{24} = 40\left(24 - \frac{576}{48}\right) = 40(24 - 12) = 480\text{ L}.$$

Second term: let $k = \frac{\pi}{12}$. Then we need

$$I = 10\int_0^{24}\sin(kt)\left(1 - \frac{t}{24}\right)dt = 10\int_0^{24}\sin(kt)\,dt - \frac{10}{24}\int_0^{24} t\sin(kt)\,dt.$$

The first integral is zero as shown in part (a):

$$\int_0^{24} \sin(kt)\,dt = \left[-\frac{1}{k}\cos(kt)\right]_0^{24} = -\frac{12}{\pi}\left[\cos(2\pi) - \cos 0\right] = 0.$$

Thus $I = -\frac{10}{24}J$ where $J = \int_0^{24} t\sin(kt)\,dt$. Evaluate J by integration by parts: Let $u = t$, $dv = \sin(kt)dt$. Then $du = dt$, $v = -\frac{1}{k}\cos(kt)$.

$$J = \left[-\frac{t}{k}\cos(kt)\right]_0^{24} + \frac{1}{k}\int_0^{24} \cos(kt)\,dt.$$

At $t = 24$, $\cos(24k) = \cos(2\pi) = 1$; at $t = 0$, term zero. So

$$J = -\frac{24}{k}\cdot 1 + \frac{1}{k}\left[\frac{1}{k}\sin(kt)\right]_0^{24} = -\frac{24}{k} + \frac{1}{k^2}\left[\sin(2\pi) - \sin 0\right] = -\frac{24}{k}.$$

Since $k = \pi/12$, $\frac{1}{k} = \frac{12}{\pi}$, so $J = -24 \times \frac{12}{\pi} = -\frac{288}{\pi}$.
Therefore

$$I = -\frac{10}{24}\left(-\frac{288}{\pi}\right) = \frac{10 \times 288}{24\pi} = \frac{2880}{24\pi} = \frac{120}{\pi}\text{ L}.$$

Thus

$$V_A^{\text{mod}} = 480 + \frac{120}{\pi}\text{ L}.$$

Numerically, $120/\pi \approx 38.197$, so total ≈ 518.2L.

$$\boxed{V_A^{\text{mod}} = 480 + \frac{120}{\pi}\text{ L} \approx 518.2\text{ L}}.$$