



Integration – Prereading

1. Integration as the Reverse of Differentiation

Differentiation gives us the rate of change of a function. Integration (or anti-differentiation) does the opposite – it recovers the original function from its derivative. If we know that $\frac{dy}{dx} = f(x)$, then

$$y = \int f(x) dx = F(x) + C,$$

where $F(x)$ is any function whose derivative is $f(x)$, and C is an arbitrary constant called the **constant of integration**.

Why does C exist? When we differentiate a function, any constant term disappears because the derivative of a constant is zero. For example, $\frac{d}{dx}(x^2) = 2x$ and $\frac{d}{dx}(x^2 + 5) = 2x$ as well. Therefore, when we reverse the process (integrate), we cannot know whether there was originally a constant term – so we must include C to account for all possibilities. This is why every indefinite integral includes an arbitrary constant.

2. The Constant of Integration and Initial Conditions

To determine the value of this arbitrary constant, we need an **initial condition** – a known value of the function at a specific point.

Example: If $\frac{dy}{dx} = 3x^2$ and we know that $y = 5$ when $x = 1$, then

$$y = x^3 + C \quad \Rightarrow \quad 5 = 1^3 + C \quad \Rightarrow \quad C = 4.$$

Hence $y = x^3 + 4$.

In kinematics, we often know the initial velocity $v(0)$ or initial position $s(0)$ to find the constants.

3. Basic Integration Rules

Just as we have rules for differentiation, we have corresponding rules for integration.

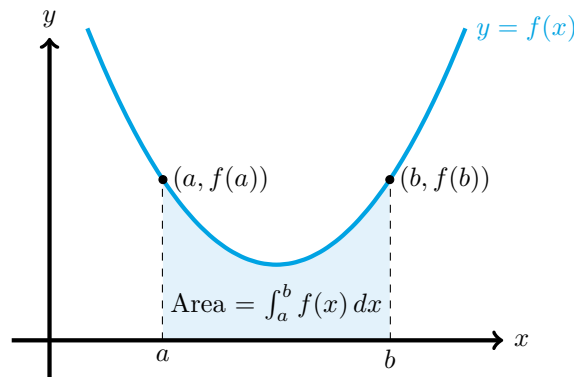
Rule Name	General Rule	Example
Power Rule	$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	$\int x^3 dx = \frac{x^4}{4} + C$
Constant Multiple Rule	$\int c f(x) dx = c \int f(x) dx$	$\int 5x^2 dx = 5 \cdot \frac{x^3}{3} + C$
Sum/Difference Rule	$\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$	$\int (x^2 + 2x) dx = \frac{x^3}{3} + x^2 + C$
Integral of a Constant	$\int a dx = ax + C$	$\int 4 dx = 4x + C$

4. Definite Integrals and Area Under a Curve

The **definite integral** $\int_a^b f(x) dx$ represents the net area under the curve $y = f(x)$ between $x = a$ and $x = b$. It is evaluated using the Fundamental Theorem of Calculus:

$$\int_a^b f(x) dx = F(b) - F(a),$$

where $F(x)$ is any anti-derivative of $f(x)$.



Example:

$$\int_0^2 3x^2 dx = [x^3]_0^2 = 2^3 - 0^3 = 8.$$

5. Kinematics: From Acceleration to Displacement

In kinematics, we use integration to move from acceleration to velocity, and from velocity to displacement.

- Given acceleration $a(t)$, velocity is $v(t) = \int a(t) dt + C_1$. The constant C_1 is found from initial velocity $v(0)$.
- Given velocity $v(t)$, displacement is $s(t) = \int v(t) dt + C_2$. The constant C_2 is found from initial displacement $s(0)$.

6. Integrating Trigonometric and Exponential Functions

Many engineering applications involve sine, cosine, and exponential functions. Here are the basic integrals:

Function	Indefinite Integral	Example
$\sin x$	$-\cos x + C$	$\int 5 \sin x dx = -5 \cos x + C$
$\cos x$	$\sin x + C$	$\int 3 \cos 2x dx = \frac{3}{2} \sin 2x + C$ (using substitution)
e^{kx}	$\frac{1}{k} e^{kx} + C$	$\int 2e^{-3t} dt = -\frac{2}{3} e^{-3t} + C$
$\frac{1}{x}$	$\ln x + C$	$\int \frac{4}{x} dx = 4 \ln x + C$

Note: For functions like $\cos 2x$ or $e^{-t/5}$, we need a simple substitution (see Section 8).

7. Advanced Integration Techniques

7.1 Integration by Substitution (Chain Rule in Reverse)

If we have an integral of the form $\int f(g(x)) g'(x) dx$, we can substitute $u = g(x)$, $du = g'(x)dx$ to simplify.

Example: $\int \cos 2x dx$. Let $u = 2x$, $du = 2dx \Rightarrow dx = du/2$. Then

$$\int \cos 2x dx = \int \cos u \cdot \frac{du}{2} = \frac{1}{2} \sin u + C = \frac{1}{2} \sin 2x + C.$$

Example with exponentials: $\int e^{-t/5} dt$. Let $u = -t/5$, $du = -\frac{1}{5}dt \Rightarrow dt = -5 du$. Then

$$\int e^{-t/5} dt = \int e^u (-5) du = -5e^u + C = -5e^{-t/5} + C.$$

7.2 Integration by Parts (Product Rule in Reverse)

The product rule for differentiation leads to the integration by parts formula:

$$\int u \, dv = uv - \int v \, du,$$

where u and v are functions of x .

Example: $\int x \cos x \, dx$. Choose $u = x$, $dv = \cos x \, dx$. Then $du = dx$, $v = \sin x$. So

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$