INSTITUTE OF
MATHEMATICS &
ITS APPLICATIONS

Integration – Problems

1. In an abuse test case, a sensor is dropped from a height of 20 m to see if its casing cracks on impact. Using integration, calculate the time taken for the sensor to hit the ground, clearly stating any physical assumptions made in your model.

2. You are an engineer designing an iron-work balcony for a café. The balcony projects 4 m from the wall and supports flower boxes along its entire length. The flower boxes create a bending moment in the beam of the form $M(x)$, where x is the distance in metres from the wall.
   The tilt (slope) of the balcony beam, $\theta(x)$, is related to the bending moment by
   $$\frac{d\theta}{dx} = \frac{M(x)}{EI},$$
   where $EI = 2400\,\text{kNm}^2$ is the beam's flexural rigidity (a measure of its resistance to bending). The bending moment (kNm) in the beam is given by:
   $$M(x) = 6x - 1.5x^2, \quad 0 \le x \le 4.$$
   (a) Write down an expression for $\frac{d\theta}{dx}$ in terms of $M(x)$ and EI.
   (b) Given that the beam is built perfectly horizontal at the wall, so $\theta(0) = 0$, determine $\theta(x)$.
   (c) Find the tilt at the free end of the balcony, $x = 4m$. Give your answer in both radians and degrees.

   The building regulations state that the maximum allowable slope for a public balcony is $2^\circ \approx 0.035\,\text{rad}$. The café owner believes that slopes greater than $1.5^\circ \approx 0.026\,\text{rad}$ are visibly noticeable and would make the flower boxes look crooked.

   (d) Using your result from part (c), decide whether the balcony satisfies the safety regulation, and whether the owner would notice the tilt just by looking at the flower boxes.
   (e) Using a graphing software to plot $\theta(x)$, justify why checking only the tilt at the free end $x = 4m$ is sufficient to answer the safety and aesthetic questions in part (d).
   (f) Prove that a higher EI (a stiffer beam) makes the tilts smaller. Briefly explain why, in practice, a designer might still choose a lower EI (a more flexible beam) even though it leads to larger tilts.

3. An aircraft's vertical acceleration, measured in ms^{-2}, during its climb to cruise altitude is modelled by:
   $$a(t) = \frac{20 - t}{10}, \quad 0 \le t \le T.$$
   Given that the initial vertical velocity $v(0) = 8\ \text{ms}^{-1}$ and the initial altitude $h(0) = 200\,\text{m}$:
   (a) Find the time T when the climb phase ends.
   (b) By determining the velocity function $v(t)$ for $0 \le t \le T$, determine the maximum vertical velocity reached.
   (c) Calculate the cruising altitude at time T.

   The rate of fuel consumption, $F(t)$ $\left(\text{kgs}^{-1}\right)$ at time t, during climb is proportional to the vertical acceleration:
   $$F(t) = 2 + 0.5a(t).$$
   (d) Find the total fuel burnt during the climb phase.

(e) Fuel efficiency, measured as the amount of fuel required to gain one meter of altitude, is a critical performance metric for aircraft during the climb phase. Determine the fuel efficiency.

4. A patient on a ventilator has their airflow rate (in litres per second) modelled by:

$$F(t) = 0.8\sin\left(\frac{\pi t}{3}\right) \quad 0 \leq t \leq 6,$$

where t is time in seconds.

(a) By sketching the function $F(t)$ over one breathing cycle, explain why the net volume over the full cycle is zero.
(b) Find the total volume of air inhaled during one 6 second cycle.
(c) Determine the average flow rate over the inhalation phase ($0 \leq t \leq 3$).
(d) The ventilator adjusts to double the frequency. Express the new flow rate function for a 3 second cycle and find the new total volume per cycle.

5. In a high storey building, an express elevator accelerates from rest with:

$$a(t) = 1 - 0.05t \quad 0 \leq t \leq 20,$$

where t is in seconds and the acceleration a has units ms^{-2}. The acceleration phase ends at $t = 20\text{s}$. Each floor is 4.0m high.

(a) Given that the initial velocity $v(0) = 0$ and the initial displacement $s(0) = 0$, find the velocity function $v(t)$ and position function $s(t)$.
(b) Calculate the velocity and height at $t = 20\text{s}$.
(c) How many floors has the elevator passed by $t = 20\text{s}$?

The elevator must stop exactly at the 40^{th} floor (160m above ground). After $t = 20\text{s}$, the elevator decelerates uniformly.

(d) Find the constant deceleration needed to stop exactly at 160m.
(e) How long does the deceleration phase last?
(f) What is the total journey time from the ground to the 40^{th} floor?

6. An electrical circuit contains a capacitor which is being charged. At time t seconds, suppose that $Q(t)$ is the charge on the capacitor and the current $I(t)$, in amps, flowing into the capacitor is given by:

$$I(t) = 3e^{-\frac{t}{5}}, \quad t \geq 0.$$

Given $I = Q'(t)$ and the initial charge on the capacitor is zero:

(a) Find the charge $Q(t)$ on the capacitor at time t.
(b) Determine the total charge that accumulates on the capacitor over the first 10 seconds.
(c) Determine the limiting charge as $t \to \infty$.
(d) Suppose the capacitor is considered "fully charged" when it reaches 95% of its limiting charge. Find the time (to 2 decimal places) when the capacitor becomes fully charged.

7. Creep is the time-dependent deformation of materials under constant stress. For a certain nickel-based superalloy used in turbine blades, suppose ϵ denotes the strain and the creep strain rate (strain per hour) is given by:

$$\frac{d\epsilon}{dt} = 0.002t^{-0.5} \quad t \geq 0,$$

where t is time in hours. At $t = 1$ hour, the strain is measured as $\epsilon = 0.001$.

(a) Find the strain $\epsilon(t)$ for $t \geq 1$.
(b) Calculate the strain and strain rate after 100 hours of operation. Express your answers as percentages.

In a laboratory test on a different material, the creep strain rate decreases exponentially with time:

$$\frac{d\epsilon}{dt} = Be^{-kt}, \quad t \geq 0$$

where B and k are positive constants. The initial strain at $t = 0$ is zero.

(c) Find an expression for $\epsilon(t)$.

(d) Experimental measurements show that when $t = 10$ hours, $\epsilon = 0.005$, and when $t = 20$ hours, $\epsilon = 0.008$. Determine the values of B and k (to 3 significant figures).

An engineer is designing a pressure vessel that will operate at high temperature for 10,000 hours. The design specification requires that the total creep strain must not exceed 0.01.

(e) Using the power-law model from part (a), determine whether the strain limit is exceeded within the 10,000-hour service life.

(f) If the limit is exceeded, calculate at what time (to the nearest hour) the strain first reaches 0.01.

8. A factory has two processes that discharge wastewater into a treatment tank. The discharge rate $r_A(t)$, in litres per hour, from Process A is modelled by:

$$r_A(t) = 40 + 10\sin\left(\frac{\pi t}{12}\right),$$

where t is the time in hours since midnight. Process B discharges at a rate $r_B(t)$, in litres per hour, given by:

$$r_B(t) = 50e^{-0.1t}, \quad t \geq 0.$$

The treatment tank has a capacity of 2000 litres and is initially empty at $t = 0$.

(a) Calculate the total volume of wastewater discharged from Process A over the first 24 hours. Then, determine the average discharge rate of Process A over this period.

(b) Find the total volume discharged from process B from $t = 0$ to $t = 24$ hours.

(c) At what time (to 2 decimal places) does the total volume discharged from process B first reach 400 litres?

(d) Suppose both processes run simultaneously from $t = 0$. Write down an expression for the combined discharge rate $R(t)$ and determine whether the tank will overflow within the first 24 hours.

In an effort to reduce discharge, the factory installs a filter on Process A. The filter reduces the discharge rate by a factor of $\left(1 - \frac{t}{24}\right)$ for $0 \leq t \leq 24$.

(e) By expressing the modified discharge rate for Process A, calculate the total volume discharged from the modified Process A over 24 hours.