EMS412U Mathematics Revision Workbook (Weeks 10-12)

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1. Matrix Algebra

A matrix is a rectangular array of numbers arranged in rows and columns. Matrices efficiently represent large systems of linear equations, which are common in engineering fields like structural analysis, circuit theory, and fluid mechanics. In this chapter, we develop the terminology and basic algebra of matrices, including how they can be added, subtracted, and multiplied.

1.1 Introduction to Matrices

A matrix is defined by its dimension, which is given by the number of rows and columns. An $m \times n$ matrix has m rows and n columns. The numbers within the matrix are called elements or entries. The element in the i-th row and j-th column of a matrix A is denoted by a_{ij}.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Special Types of Matrices

Column and Row Vectors:
A matrix with only one column is called a column vector (size $m \times 1$). A matrix with only one row is called a row vector (size $1 \times n$).

$$\mathbf{c} = \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} 4 & 0 & 9 \end{pmatrix}$$

Square Matrix:
A matrix is square if the number of rows equals the number of columns ($m = n$). The elements form a leading diagonal.

$$A = \begin{pmatrix} 2 & 5 \\ -3 & 8 \end{pmatrix}$$

Diagonal Matrix:
A square matrix where all non-zero elements are along the leading diagonal ($d_{ij} = 0$ when $i \neq j$).

$$D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Identity Matrix (I):
A diagonal matrix where all diagonal elements are 1. It plays the role of the number 1 in matrix multiplication.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Symmetric Matrix:
A matrix A is symmetric if it is equal to its transpose, i.e., $A = A^T$. Notice the symmetry across the diagonal.

$$S = \begin{pmatrix} 1 & \mathbf{7} & \mathbf{3} \\ \mathbf{7} & 4 & \mathbf{-5} \\ \mathbf{3} & \mathbf{-5} & 0 \end{pmatrix}$$

Triangular Matrices:
An Upper Triangular matrix has all elements below the leading diagonal as zero. A Lower Triangular matrix has all elements above the leading diagonal as zero.

$$U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$$

Example 1.1 Classifying Matrices

Problem: Classify the following matrices by their type and dimensions:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution:

- Matrix A has 2 rows and 3 columns. It is a 2×3 matrix (rectangular).
- Matrix B has 3 rows and 3 columns. It is a 3×3 square matrix. Since all elements above the diagonal are zero, it is a lower triangular matrix.
- Matrix C is a 3×3 square matrix. All non-diagonal elements are zero, and diagonal elements are 1. It is the Identity matrix (I_3).

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Q1. Given the matrix $A = \begin{pmatrix} 1 & 4 & 6 & 6 \\ 0 & 4 & 2 & 6 \end{pmatrix}$:

(a) State the dimension of A.
(b) Find the values of elements $A_{1,2}$ and $A_{2,3}$.

[**ans:** $(a)\ 2 \times 4, \quad (b)\ 4,\ 2$]

Q2. Which of the following matrices are symmetric?

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

[**ans:** C only]

Q3. Identify whether the following matrices are Upper Triangular, Lower Triangular, or neither:

$$U = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 3 & 9 \\ 0 & 0 & 5 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 9 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 1 & 0 & 5 \end{pmatrix}$$

[**ans:** U : Upper, L : Lower, N : Neither]

1.2 Matrix Arithmetic

1.2.1 Addition and Subtraction

Two matrices A and B can only be added or subtracted if they have the same dimensions (same number of rows and same number of columns). The sum $C = A + B$ is found by adding corresponding elements:

$$c_{ij} = a_{ij} + b_{ij}$$

1.2.2 Scalar Multiplication

A matrix A can be multiplied by a scalar (number) k. The result kA is obtained by multiplying every element of A by k.

$$kA = \begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{pmatrix}$$

1.2.3 The Transpose

The transpose of a matrix A, denoted A^T, is formed by swapping the rows and columns of A. If A is an $m \times n$ matrix, A^T is an $n \times m$ matrix.

$$\text{If } A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \text{ then } A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

Example 1.2 Matrix Operations Part 1

Problem: Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$, calculate $2A - B$.

Solution:

Step 1: Multiply matrix A by the scalar 2.

$$2A = 2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

Step 2: Subtract matrix B from $2A$.

$$2A - B = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} - \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

Step 3: Subtract corresponding elements.

$$= \begin{pmatrix} 2-5 & 4-6 \\ 6-7 & 8-8 \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ -1 & 0 \end{pmatrix}$$

$$\boxed{2A - B = \begin{pmatrix} -3 & -2 \\ -1 & 0 \end{pmatrix}}$$

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Example 1.3 Matrix Operations Part 2

Problem: Given the matrices $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}$, find:

1. $A + B$
2. $2A - B^T$

Solution:

1. Addition:

$$A + B = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1+4 & 2+1 \\ 3+0 & 0+2 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

2. Scalar Multiplication and Transpose:

$$2A = \begin{pmatrix} 2(1) & 2(2) \\ 2(3) & 2(0) \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 0 \end{pmatrix}$$

$$B^T = \begin{pmatrix} 4 & 0 \\ 1 & 2 \end{pmatrix} \quad \text{(swapping rows and columns)}$$

$$2A - B^T = \begin{pmatrix} 2 & 4 \\ 6 & 0 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 5 & -2 \end{pmatrix}$$

$$\boxed{2A - B^T = \begin{pmatrix} -2 & 4 \\ 5 & -2 \end{pmatrix}}$$

■

Q1. Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, calculate:

(a) $A+B$
(b) $2A$
(c) $B-A$

$\left[\textbf{ans:}\ (a)\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix},\ (b)\begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix},\ (c)\begin{pmatrix} 0 & -1 \\ -2 & -3 \end{pmatrix}\right]$

Q2. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 5 \\ 6 & 1 \\ 4 & 3 \end{pmatrix}$, explain why $A+B$ is not defined.

[**ans:** Matrices have different dimensions (2×3 vs 3×2).]

Q3. Given the matrices below, verify that $(A+B)^T = A^T + B^T$.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Q4. Find the matrix X such that $A + X = B$, given:

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 10 & 8 \\ 4 & 6 \end{pmatrix}$$

$\left[\textbf{ans:}\ X = \begin{pmatrix} 8 & 3 \\ 3 & 3 \end{pmatrix}\right]$

Q5. Given $A = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}$, find the matrix equal to $3A^T$.

$\left[\textbf{ans:}\ \begin{pmatrix} 3 & -6 \\ 0 & 9 \end{pmatrix}\right]$

1.3 Matrix Multiplication

Multiplying matrices is different from multiplying numbers. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, the product AB is defined and results in an $m \times p$ matrix. The number of columns in A must equal the number of rows in B.

1.3.1 The Multiplication Rule

The element in the i-th row and j-th column of the product $C = AB$ is obtained by multiplying the row i of A by the column j of B and summing the products.

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

1.3.2 Properties of Multiplication

Unlike numbers, matrix multiplication is not commutative. In general, $AB \neq BA$. However, the associative law, $A(BC) = (AB)C$ and distributive law, $A(B+C) = AB + AC$ still hold.

Example 1.4 Matrix Multiplication (2×2 **times** 2×3)

Problem: Calculate AB where $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \end{pmatrix}$.

Solution:

Step 1: Check dimensions. A is (2×2), B is (2×3).
The inner numbers match (2), so the result will be size 2×3.

Step 2: Calculate Row 1 of the result (using Row 1 of A).

Col 1: $(1)(2) + (2)(-1) = 2 - 2 = 0$

Col 2: $(1)(0) + (2)(3) = 0 + 6 = 6$

Col 3: $(1)(1) + (2)(2) = 1 + 4 = 5$

Step 3: Calculate Row 2 of the result (using Row 2 of A).

Col 1: $(3)(2) + (4)(-1) = 6 - 4 = 2$

Col 2: $(3)(0) + (4)(3) = 0 + 12 = 12$

Col 3: $(3)(1) + (4)(2) = 3 + 8 = 11$

$$\boxed{AB = \begin{pmatrix} 0 & 6 & 5 \\ 2 & 12 & 11 \end{pmatrix}}$$

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Q1. Given $A = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 4 & 1 \\ 6 & 1 & 0 \end{pmatrix}$.

(a) Determine the size of the product AB.
(b) Calculate AB.

$\left[\textbf{ans: } (a)\ 2 \times 3, \quad (b) \begin{pmatrix} -10 & 2 & 1 \\ -14 & 5 & 2 \end{pmatrix}\right]$

Q2. Calculate AB and BA where $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. What do you notice about the results?

$\left[\textbf{ans: } AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right]$

Q3. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Show that $AI = A$.

Q4. If $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$, calculate A^2 (which is $A \times A$).

$\left[\textbf{ans: } \begin{pmatrix} 4 & 5 \\ 0 & 9 \end{pmatrix}\right]$

Chapter Review

Review Exercise 1 Mixed Questions

Problem 1.3.1 Given the matrices:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 1 & 3 \\ 0 & 2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{pmatrix}$$

Which of the following operations are possible?

(a) $A+B$
(b) $A+C$
(c) AB
(d) AC
(e) CB

Problem 1.3.2 If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 4 \\ 0 & 1 \\ 2 & 7 \end{pmatrix}$, find the matrix $3A - 3B^T$.

Problem 1.3.3 Given $A = \begin{pmatrix} 2 & 3 & -1 \\ 0 & 1 & 2 \\ 4 & 5 & 6 \end{pmatrix}$, show that the matrix AA^T is symmetric.

Problem 1.3.4 Identify which of the following matrices are square, lower triangular, upper triangular, or diagonal:

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 1.3.5 Verify the distributive law $A(B+C) = AB + AC$ using the following matrices:

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

Problem 1.3.6 Find the values of x and y given the following matrix equality:

$$\begin{pmatrix} x+y & 2 \\ 4 & x-y \end{pmatrix} = \begin{pmatrix} 8 & 2 \\ 4 & 2 \end{pmatrix}$$

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Summary 1 Matrices

Matrix Dimensions An $m \times n$ matrix has m rows and n columns.

Special Matrices

- **Square:** Same number of rows and columns.
- **Identity (I):** Diagonal matrix with 1s on the diagonal.
- **Symmetric:** $A = A^T$.

Transposition The transpose A^T is found by swapping rows and columns.

Addition Possible only if matrices have the same dimensions. $A+B = B+A$.

Multiplication The product AB exists only if the number of columns in A equals the number of rows in B. In general, $AB \neq BA$.

Chapter Checklist

- ☐ I can identify the dimensions of a matrix and specific elements.
- ☐ I can classify matrices as square, diagonal, identity, or symmetric.
- ☐ I can perform matrix addition, subtraction, and scalar multiplication.
- ☐ I can find the transpose of a matrix.
- ☐ I understand that matrix multiplication is generally not commutative.

2. Determinants

A determinant is a specific number associated with a square matrix. Unlike a matrix, which is an array of numbers, the determinant is a single scalar value. It provides important information about the matrix, such as whether the matrix has an inverse and whether a system of linear equations has a unique solution. The determinant of a matrix A is denoted by $\det(A)$, $|A|$, or Δ.

2.1 Evaluating Determinants

2.1.1 The 2×2 Case

For a 2×2 matrix, the determinant is calculated by finding the product of the leading diagonal elements and subtracting the product of the other diagonal elements.

$$\text{If } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{then } |A| = ad - bc$$

Example 2.1 **Calculating a 2×2 Determinant**

Problem: Calculate the determinant of $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Solution:

$$\begin{aligned} |A| &= (1)(4) - (2)(3) \\ &= 4 - 6 \\ &= -2 \end{aligned}$$

$$\boxed{|A| = -2}$$

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2.1.2 The 3×3 Case (Laplace Expansion)

To evaluate determinants of order 3 or higher, we often use the method of Laplace expansion (or expansion by minors). This involves breaking the determinant down into smaller determinants using minors and cofactors.

- **Minor (M_{ij}):** The minor of an element a_{ij} is the determinant of the sub-matrix left after removing row i and column j.
- **Cofactor (A_{ij}):** The cofactor of an element is its minor with a sign attached, based on the element's position. The sign pattern for a 3×3 matrix alternates:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

- Mathematically, the cofactor is $A_{ij} = (-1)^{i+j} M_{ij}$.

The determinant is found by multiplying each element of any single row (or column) by its corresponding cofactor and summing the results. Expanding along the first row gives:

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example 2.2 Calculating a 3×3 Determinant

Problem: Evaluate $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 8 & 1 \end{vmatrix}$ by expanding along the first row.

Solution:

$$\Delta = 1\begin{vmatrix} 5 & 4 \\ 8 & 1 \end{vmatrix} - 2\begin{vmatrix} 6 & 4 \\ 7 & 1 \end{vmatrix} + 3\begin{vmatrix} 6 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(5-32) - 2(6-28) + 3(48-35)$$

$$= 1(-27) - 2(-22) + 3(13)$$

$$= -27 + 44 + 39$$

$$= 56$$

$$\boxed{\Delta = 56}$$

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Q1. Evaluate the determinant of $B = \begin{pmatrix} 4 & -1 \\ -2 & -3 \end{pmatrix}$. [**ans:** -14]

Q2. Calculate the minor and cofactor of the element a_{22} (the center element) for the matrix:

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

[**ans:** Minor $= 5$, Cofactor $= +5$]

Q3. Use expansion along the first row to determine $|A|$ for:

$$A = \begin{pmatrix} 3 & 1 & -4 \\ 6 & 9 & -2 \\ -1 & 2 & 1 \end{pmatrix}$$

[**ans:** -49]

2.2 Properties of Determinants

Evaluating determinants for large matrices can be laborious. Several properties can help simplify the calculation.

1. **Identity Matrix:** $|I| = 1$.
2. **Transpose:** The determinant of a matrix and its transpose are equal: $|A| = |A^T|$.
3. **Row Exchange:** Swapping two rows (or columns) reverses the sign of the determinant.
4. **Identical Rows/Columns:** If two rows (or columns) are identical, the determinant is zero.
5. **Zero Row/Column:** If a matrix has a row (or column) of all zeros, the determinant is zero.
6. **Scalar Multiplication (Row):** If a single row is multiplied by a constant k, the determinant is multiplied by k.
   Note: If the entire $n \times n$ matrix is multiplied by k, then $|kA| = k^n |A|$.
7. **Row Operations:** Adding a multiple of one row to another does **not** change the value of the determinant. This is very useful for introducing zeros to simplify expansion.
8. **Triangular Matrices:** The determinant of an upper or lower triangular matrix (or a diagonal matrix) is simply the product of the diagonal elements.

   $$|A| = a_{11} \times a_{22} \times \cdots \times a_{nn}$$

9. **Matrix Product:** The determinant of a product is the product of the determinants: $|AB| = |A||B|$.

Example 2.3 Using Properties to Evaluate a Determinant

Problem: Evaluate the determinant using row operations to simplify:

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{vmatrix}$$

Solution:

Step 1: Use row operations to create zeros in the first column.

Perform $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$.

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & -3 & -1 \end{vmatrix}$$

Step 2: Continue row reduction to make it upper triangular.

Perform $R_3 \to R_3 - 3R_2$.

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -4 \end{vmatrix}$$

Step 3: Calculate the product of the diagonal elements.

$\Delta = 1 \times (-1) \times (-4) = 4$

$\boxed{\Delta = 4}$

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Q1. Without expanding, state the value of the following determinant and explain why:

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

[**ans:** 0 (Two rows are identical)]

Q2. Evaluate the determinant of the diagonal matrix $E = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$. [**ans:** 8]

Q3. If A is a 3×3 matrix with $|A| = 4$, find the value of $|2A|$.

[**ans:** $2^3|A| = 8 \times 4 = 32$]

Q4. Calculate the determinant of the following upper triangular matrix:

$$T = \begin{pmatrix} 5 & 9 & 2 \\ 0 & -2 & 4 \\ 0 & 0 & 3 \end{pmatrix}$$

[**ans:** $5 \times (-2) \times 3 = -30$]

Chapter Review

Review Exercise 2 Mixed Questions

Problem 2.2.1 Evaluate the determinants of the following matrices:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

Problem 2.2.2 Find the values of λ that make the determinant zero:

$$\begin{vmatrix} 2-\lambda & 7 \\ 4 & 6-\lambda \end{vmatrix} = 0$$

Problem 2.2.3 Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -1 \\ -2 & -3 \end{pmatrix}$, show that $|AB| = |A||B|$.

Problem 2.2.4 Evaluate the determinant using properties:

$$\begin{vmatrix} 12 & 27 & 12 \\ 28 & 18 & 24 \\ 70 & 15 & 40 \end{vmatrix}$$

Problem 2.2.5 Find the value of k for which the matrix M is singular (i.e., $|M| = 0$):

$$M = \begin{pmatrix} 1 & 2 & 1 \\ 2 & k & 2 \\ 3 & 6 & 3 \end{pmatrix}$$

Problem 2.2.6 Let $|A| = 5$. If matrix B is obtained from matrix A by adding 3 times Row 1 to Row 2, what is the value of $|B|$?

Problem 2.2.7 Given that the determinant of matrix P is 7, what is the value of $|P^T|$?

Problem 2.2.8 If I_3 is the 3×3 identity matrix, evaluate the determinant $|4I_3|$.

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Summary 2 Determinants

Calculation

- 2×2 **Matrix:** $|A| = ad - bc$
- 3×3 **Matrix:** Calculate using Laplace expansion with minors and cofactors.

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

Key Properties

- If a row or column is zero, $|A| = 0$.
- If two rows or columns are identical, $|A| = 0$.
- Swapping rows changes the sign: $|B| = -|A|$.
- The determinant of a triangular matrix is the product of its diagonal elements.
- $|A^T| = |A|$ and $|AB| = |A||B|$.

Chapter Checklist

- ☐ I can calculate the determinant of a 2×2 matrix.
- ☐ I can find the minors and cofactors of a matrix.
- ☐ I can calculate a 3×3 determinant using Laplace expansion.
- ☐ I can use row operations to simplify a matrix before calculating its determinant.
- ☐ I can solve for unknown variables (like λ) inside a determinant.

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3. Inverse Matrices and Linear Equations

Just as numbers have reciprocals (e.g., the reciprocal of 8 is $1/8$), square matrices can have inverses. The inverse of a matrix A is denoted A^{-1} and satisfies the property $AA^{-1} = A^{-1}A = I$. This concept is powerful because it allows us to solve systems of linear equations in the form $Ax = b$ algebraically, similar to how we solve $ax = b$ by dividing by a. In this chapter, we explore methods to calculate the inverse and techniques to solve linear systems, including Gaussian elimination.

3.1 The Inverse of a Matrix

An $n \times n$ square matrix A is called invertible or non-singular if its determinant is non-zero ($|A| \neq 0$). If $|A| = 0$, the matrix is singular and has no inverse. If an inverse exists, it is unique.

3.1.1 Method 1: The Adjoint Method

For a matrix A, the inverse can be calculated using the determinant and the adjoint matrix (adj A). The adjoint matrix is the transpose of the matrix of cofactors.

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, this simplifies to:

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

3.1.2 Method 2: Gauss-Jordan Elimination

For larger matrices, finding the inverse using cofactors can be tedious. An alternative is to use row operations. We set up an augmented matrix $[A|I]$ and perform row operations to transform A into the identity matrix I. The right side will simultaneously transform into A^{-1}:

$$[A|I] \xrightarrow{\text{row operations}} [I|A^{-1}]$$

This works because the operations that turn A into I are equivalent to multiplying by A^{-1}.

Example 3.1 **Finding the Inverse of a 3×3 Matrix**

Problem: Find the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 8 & 1 \end{pmatrix}$ using the adjoint method.

Solution:

Step 1: Calculate the determinant.

$$|A| = 1(5-32) - 2(6-28) + 3(48-35)$$
$$|A| = -27 + 44 + 39 = 56 \quad (\text{Since } |A| \neq 0, A^{-1} \text{ exists}).$$

Step 2: Find the matrix of cofactors.

$$C_{11} = +\begin{vmatrix} 5 & 4 \\ 8 & 1 \end{vmatrix} = -27, \quad C_{12} = -\begin{vmatrix} 6 & 4 \\ 7 & 1 \end{vmatrix} = 22, \quad \ldots$$

Completing this for all elements, we get the cofactor matrix:

$$C = \begin{pmatrix} -27 & 22 & 13 \\ 22 & -20 & 6 \\ -7 & 14 & -7 \end{pmatrix}$$

Step 3: Transpose to find the adjoint and multiply by $1/|A|$.

$$\text{adj } A = C^T = \begin{pmatrix} -27 & 22 & -7 \\ 22 & -20 & 14 \\ 13 & 6 & -7 \end{pmatrix}$$

$$\boxed{A^{-1} = \frac{1}{56}\begin{pmatrix} -27 & 22 & -7 \\ 22 & -20 & 14 \\ 13 & 6 & -7 \end{pmatrix}}$$

■

Q1. Find the inverse of the matrix $A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$. $\left[\textbf{ans: } \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}\right]$

Q2. Given $A = \begin{pmatrix} 1 & 0 & 2 \\ 6 & 4 & 0 \\ 6 & -2 & 1 \end{pmatrix}$, calculate A^{-1}.

$$\left[\textbf{ans: } -\tfrac{1}{68}\begin{pmatrix} 4 & -4 & -8 \\ -6 & -11 & 12 \\ -36 & 2 & 4 \end{pmatrix}\right]$$

Q3. Verify that $(AB)^{-1} = B^{-1}A^{-1}$ for the matrices in Q1 and Q2.

3.2 Solving Systems of Linear Equations

A system of linear equations can be written in the matrix form $Ax = b$, where A is the matrix of coefficients, x is the column vector of unknowns, and b is the column vector of constants.

$$\begin{cases} x - 2y = 0 \\ 2x - y = 3 \end{cases} \implies \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

Solution Types

1. **Unique Solution:** If $|A| \neq 0$, the inverse exists, and the unique solution is $x = A^{-1}b$. Geometrically, this represents lines or planes intersecting at a single point.
2. **No Solution:** If $|A| = 0$ and the equations are inconsistent (e.g., parallel lines).
3. **Infinite Solutions:** If $|A| = 0$ and the equations are consistent (e.g., identical lines).

Example 3.2 Solving Linear Equations using the Inverse Matrix

Problem: Solve the system:

$$\begin{cases} x - 2y = 0 \\ 2x - y = 3 \end{cases}$$

Solution:

Step 1: Identify A and b.

$$A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

Step 2: Find A^{-1}.

$$|A| = (1)(-1) - (-2)(2) = 3$$

$$A^{-1} = \frac{1}{3} \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$$

Step 3: Calculate $x = A^{-1}b$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\boxed{x = 2, \quad y = 1}$$

■

3.3 Gaussian Elimination

For larger systems, calculating the inverse is inefficient. Gaussian elimination is a more robust method. We form an augmented matrix $[A|b]$ and perform row operations to reach row-echelon form. Interpreting the result:

- **Unique Solution:** The final matrix gives explicit values for variables (e.g., $z = -2$, then back-substitute).
- **No Solution (Inconsistent):** A row appears like $(0\ 0\ 0 \mid k)$ where $k \neq 0$. This implies $0 = k$, which is impossible.
- **Infinite Solutions:** A row of zeros appears $(0\,0\,0 \mid 0)$. This implies one variable is free (a parameter, usually t), and others are expressed in terms of it.

Example 3.3 Infinite Solutions (Parameterisation)

Problem: Solve the system:

$$\begin{cases} x+y-z=1 \\ 2x-y+2z=2 \\ -3y+4z=0 \end{cases}$$

Step 1: Form the augmented matrix.

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & -1 & 2 & 2 \\ 0 & -3 & 4 & 0 \end{pmatrix}$$

Step 2: Perform row operations ($R_2 \to R_2 - 2R_1$).

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & -3 & 4 & 0 \\ 0 & -3 & 4 & 0 \end{pmatrix}$$

Step 3: Eliminate further ($R_3 \to R_3 - R_2$).

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 4: Interpret the result.

Row 3 is all zeros, so we have infinite solutions. Let $z = t$.

From Row 2: $-3y + 4t = 0 \implies y = \frac{4}{3}t$.

From Row 1: $x + \frac{4}{3}t - t = 1 \implies x = 1 - \frac{1}{3}t$.

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1/3 \\ 4/3 \\ 1 \end{pmatrix}}$$

■

Q1. Solve the system, or show that it has no solution:

$$\begin{cases} 2x-5y+3z=8 \\ 3x-y+4z=7 \\ x+3y+2z=-3 \end{cases}$$

[**ans:** (6,-1,3)]

Q2. Solve the system, or show that it has no solution:

$$\begin{cases} x+y-z=1 \\ 2x-y+2z=2 \\ -3y+4z=3 \end{cases}$$

[**ans:** No Solution (Inconsistent)]

Q3. Solve the system, or show that it has no solution:

$$\begin{cases} x-y-z=1 \\ -x+2y-3z=-4 \\ 3x-2y-7z=0 \end{cases}$$

[**ans:** Infinitely many solutions (x,y,z), where $x=5z-2, y=4z-3$, z is any real number.]

Chapter Review

Review Exercise 3 Mixed Questions

Problem 3.3.1 Find the inverse of $A=\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and use it to solve $Ax=\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

Problem 3.3.2 Use Gauss elimination to find the solution:

$$\begin{cases} x_2+x_3=1 \\ x_1+3x_3+2x_4=3 \\ 2x_1+x_2+5x_3+4x_4=7 \end{cases}$$

Problem 3.3.3 For what value of λ does the system have no unique solution? (Hint: check where $|A|=0$).

$$\begin{cases} 2x+\lambda y=0 \\ 4x+6y=0 \end{cases}$$

Problem 3.3.4 Given $A=\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$, find A^{-1} and verify explicitly that $AA^{-1}=I$.

■

Summary 3 Inverse Matrices and Linear Equations

Inverse Matrix An $n \times n$ matrix A has an inverse A^{-1} if and only if $|A| \neq 0$.

Calculating Inverses

- **Adjoint Method:** $A^{-1} = \frac{1}{|A|}\text{adj}(A)$.
- **Row Operations:** Reduce $[A|I]$ to $[I|A^{-1}]$.

Linear Systems A system $Ax = b$ can be solved using $x = A^{-1}b$ (if $|A| \neq 0$).

Gaussian Elimination A universal method for solving systems.

- Consistent and independent equations → Unique solution.
- Consistent and dependent equations → Infinite solutions (parameterize).
- Inconsistent equations → No solution.

Chapter Checklist

- ☐ I can calculate the inverse of a matrix using the adjoint method.
- ☐ I can calculate the inverse using Gauss-Jordan elimination.
- ☐ I can solve systems of linear equations using the inverse matrix.
- ☐ I can interpret the geometric meaning of solutions (intersection of planes).
- ☐ I can identify when a system has unique, infinite, or no solutions.

■

4. Eigenvalues and Eigenvectors

When a matrix A acts on a vector v (by multiplication Av), it typically rotates and stretches that vector. However, for certain special non-zero vectors, the matrix action acts only as a scaling factor, keeping the vector's direction unchanged. These special vectors are called eigenvectors, and the scaling factors are called eigenvalues. This concept is fundamental in engineering for vibration analysis, stability theory, and transforming matrices into simpler diagonal forms.

4.1 Definitions and Calculation

An eigenvector of a square matrix A is a non-zero vector v such that:

$$Av = \lambda v$$

where λ is a scalar called the eigenvalue.

4.1.1 Finding Eigenvalues

To find the eigenvalues, we rearrange the equation to $(A - \lambda I)v = 0$. For a non-trivial solution ($v \neq 0$) to exist, the matrix $(A - \lambda I)$ must be singular. Therefore, we solve the characteristic equation:

$$\det(A - \lambda I) = 0$$

The roots of this polynomial equation are the eigenvalues.

4.1.2 Finding Eigenvectors

Once an eigenvalue λ is found, we substitute it back into the equation $(A - \lambda I)v = 0$ and solve the resulting linear system to find the corresponding eigenvector v.

Example 4.1 Calculating Eigenvalues and Eigenvectors

Problem: Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 4 & 2 \\ 5 & 7 \end{pmatrix}$.

Solution:

Step 1: Solve the characteristic equation $|A-\lambda I|=0$.

$$\begin{vmatrix} 4-\lambda & 2 \\ 5 & 7-\lambda \end{vmatrix} = (4-\lambda)(7-\lambda)-10=0$$

$$\lambda^2-11\lambda+28-10=0$$

$$\lambda^2-11\lambda+18=0$$

$$(\lambda-9)(\lambda-2)=0 \implies \lambda_1=9,\ \lambda_2=2.$$

Step 2: Find the eigenvector for $\lambda_1=9$.

$$(A-9I)v=\begin{pmatrix} -5 & 2 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-5x+2y=0 \implies 2y=5x. \text{ Let } x=2, \text{ then } y=5.$$

$$v_1=\begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

Step 3: Find the eigenvector for $\lambda_2=2$.

$$(A-2I)v=\begin{pmatrix} 2 & 2 \\ 5 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2x+2y=0 \implies y=-x. \text{ Let } x=1, \text{ then } y=-1.$$

$$v_2=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\boxed{\lambda_1=9,\ v_1=\begin{pmatrix} 2 \\ 5 \end{pmatrix}; \quad \lambda_2=2,\ v_2=\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

■

Q1. Find the eigenvalues of the matrix $A=\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$. [**ans:** $\lambda=1,2$]

Q2. Find the eigenvectors corresponding to the eigenvalues found in Q1.

[**ans:** For $\lambda=1, v=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$; for $\lambda=2, v=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$]

Q3. Calculate the eigenvalues of $A=\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

[**ans:** $\lambda=2,\ 2-\sqrt{3},\ 2+\sqrt{3}$]

Q4. Determine whether the vector $v=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix $A=\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$. If it is, state the corresponding eigenvalue.

[**ans:** $Av=\begin{pmatrix} 5 \\ 5 \end{pmatrix}=5v$. Yes, $\lambda=5$.]

4.2 Properties and Diagonalisation

4.2.1 Properties of Eigenvalues

1. **Trace:** The sum of the eigenvalues equals the trace of the matrix (sum of diagonal elements).

$$\sum \lambda_i = \text{tr}(A)$$

2. **Determinant:** The product of the eigenvalues equals the determinant of the matrix.

$$\prod \lambda_i = \det(A)$$

3. **Linearly Independent:** Eigenvectors corresponding to distinct eigenvalues are linearly independent.

4. **Symmetric Matrices:** If A is symmetric, its eigenvalues are always real, and eigenvectors corresponding to distinct eigenvalues are orthogonal ($v_1^T v_2 = 0$).

4.2.2 Diagonalisation

If an $n \times n$ matrix A has n linearly independent eigenvectors, it can be diagonalised. We form a modal matrix P where the columns are the eigenvectors. Then:

$$P^{-1}AP = D$$

where D is a diagonal matrix containing the eigenvalues. This is particularly useful for calculating powers of matrices, as $A^k = PD^kP^{-1}$.

Example 4.2 Diagonalising a Matrix

Problem: Given $A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ with eigenvalues $\lambda_1 = -1, \lambda_2 = 5$ and eigenvectors $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Find P and D.

Solution:

Step 1: Construct the modal matrix P from the eigenvectors.

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Step 2: Construct the diagonal matrix D from the eigenvalues.

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$$

Step 3: Verify $P^{-1}AP = D$.

$$P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$P^{-1}AP = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$$

$$\boxed{D = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}}$$

■

Q1. Verify the trace and determinant properties for $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ with $\lambda = 2, 2 \pm \sqrt{2}$.

(a) Sum of eigenvalues vs Trace.
(b) Product of eigenvalues vs Determinant.

$[\textbf{ans: } (a)\ 6 = 6, \quad (b)\ 4 = 4]$

Q2. If $D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$, calculate D^3. $\left[\textbf{ans: } \begin{pmatrix} 27 & 0 \\ 0 & -8 \end{pmatrix}\right]$

Q3. For the symmetric matrix $A = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$, find the eigenvalues and show that the eigenvectors are orthogonal. $[\textbf{ans: } \lambda = 0, 5;\ v_1 \cdot v_2 = 0]$

Q4. Without calculating the characteristic equation, state the eigenvalues of the upper triangular matrix:

$$T = \begin{pmatrix} 5 & 1 & 7 \\ 0 & -3 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

$[\textbf{ans: } \lambda = 5, -3, 4 \text{ (diagonal elements)}]$

Q5. A matrix A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$. Find the eigenvalues of the matrix A^5.
$[\textbf{ans: } \lambda_1^5 = 1^5 = 1, \quad \lambda_2^5 = 2^5 = 32]$

Chapter Review

Review Exercise 4 Mixed Questions

Problem 4.2.1 Given the matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$:

(a) Find the eigenvalues and eigenvectors of A.
(b) Find the eigenvalues of A^2 and A^{-1} without full calculation.

Problem 4.2.2 Prove that if λ is an eigenvalue of A, then $\lambda + k$ is an eigenvalue of $A + kI$.

Problem 4.2.3 A system of differential equations is given by $\dot{x} = 4x + 2y$ and $\dot{y} = -x + y$. Using matrix diagonalisation, find the general solution.

Problem 4.2.4 A 2×2 matrix A has a determinant of 6 and a trace of 5.
(a) Find the two eigenvalues of A.
(b) If one eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, find the matrix A assuming it is symmetric.

■

Summary 4 Eigenvalues and Eigenvectors

Definition For a square matrix A, if $Av = \lambda v$ ($v \neq 0$), then λ is the eigenvalue and v is the eigenvector.

Calculation

- **Eigenvalues:** Solve $\det(A - \lambda I) = 0$.
- **Eigenvectors:** Solve $(A - \lambda I)v = 0$.

Properties

- Sum of eigenvalues = Trace(A).
- Product of eigenvalues = Det(A).
- Eigenvalues of A^k are λ^k.
- Symmetric matrices have real eigenvalues and orthogonal eigenvectors.

Diagonalisation $A = PDP^{-1}$, where D holds the eigenvalues and P holds the eigenvectors.

Chapter Checklist

- ☐ I can calculate eigenvalues by solving the characteristic equation.
- ☐ I can find eigenvectors for a given eigenvalue.
- ☐ I can verify properties using the trace and determinant.
- ☐ I can construct the modal matrix P and diagonal matrix D.
- ☐ I understand the properties of symmetric matrices.