EMS412U Mathematics Revision Workbook (Weeks 1-4)

Student Co-Creators: Abul Hassan Mohammed Ibrahim, Ali Mahmood, Deehan Haque, Morshid Sarker and Saeed Ahmed

Contact email: a.h.mohammedibrahim@se24.qmul.ac.uk

Academic Lead: Dr. Rehan Shah

Contents

1. Functions

1.1 What is a function?

A function is a rule which operates on an input and produces a single output from that input. The input to a function is sometimes called its argument. It is frequently necessary to obtain the output from a function if we are given its argument.

Example 1.1 Function Evaluation

Problem: Given the function $g(x) = \dfrac{2x^2-3}{(x+1)^2}$, find the value of $g(3)$.

Solution: To find $g(3)$, we substitute $x = 3$ into the expression for the function:

$$\begin{aligned} g(3) &= \frac{2(3)^2-3}{(3+1)^2} \\ &= \frac{2(9)-3}{4^2} \\ &= \frac{18-3}{16} \\ &= \frac{15}{16} \end{aligned}$$

Final Answer:

$$\boxed{g(3) = \frac{15}{16}}$$

■

Q1. Given the function $y(x) = 3x+2$, find:

(a) $y(3)$ **[ans:** 11]

(b) $y(2t)$ **[ans:** 6t+2]

(c) $y(z+2)$ **[ans:** 3z+8]

Q2. Calculate $f(x+h)$ and find the corresponding expression for $f(x+h)-f(x)$ for the following functions:

(a) $f(x)=x^2$ $\left[\textbf{ans: } x^2+2xh+h^2, \quad 2xh+h^2\right]$

(b) $f(x)=x^3$ $\left[\textbf{ans: } x^3+3x^2h+3xh^2+h^3, \quad 3x^2h+3xh^2+h^3\right]$

Q3. If $f(x)=\dfrac{3x+1}{(x-1)^2}$, find $f\left(\dfrac{3z}{l}\right)$. $\left[\textbf{ans: } \dfrac{l(9z+l)}{(3z-l)^2}\right]$

Q4. The function f is given by

$$f: x \mapsto \frac{x}{x+3}, \quad x \in \mathbb{R},\, x \neq -3.$$

The function g is defined as

$$g: x \mapsto \frac{2}{x}, \quad x \in \mathbb{R},\, x \neq 0.$$

(a) Find an expression for $f^{-1}(x)$. $\left[\textbf{ans: } \dfrac{-3x}{x-1}, \quad x \neq 1\right]$

(b) Find an expression for $g^{-1}(x)$. $\left[\textbf{ans: } \dfrac{2}{x}, \quad x \neq 0\right]$

Q5. The function f is given by

$$f(x)=\ln(4x-2), \quad x \in \mathbb{R},\, x > \frac{1}{2}.$$

(a) Find an expression for $f^{-1}(x)$, in its simplest form. $\left[\textbf{ans: } \dfrac{e^x+2}{4}\right]$

(b) State the range of $f^{-1}(x)$. $\left[\textbf{ans: } \left(\dfrac{1}{2}, \infty\right)\right]$

(c) Solve the equation: $f(x)=1$. $\left[\textbf{ans: } \dfrac{e+2}{4}\right]$

Q6. Evaluate $f(3x)$ for the following function where $x<0$:

$$f(x)=\begin{cases} x^3+x^2-2, & \text{if } x<0 \\ 2x, & \text{if } x \geq 0 \end{cases}$$

$\left[\textbf{ans: } 27x^3+9x^2-2 \quad \text{for } x<0\right]$

Q7. The function f is given by:

$$f(x)=3-\ln x, \quad x \in \mathbb{R},\, x>0.$$

Given this, solve the equation $f(x)=4$. $\left[\textbf{ans: } \dfrac{1}{e}\right]$

1.2 Domain and range of a function

The set of values which we allow the independent variable to take is called the domain of the function. A domain is often an interval on the x axis. The set of values of the function for a given domain, that is, the set of y values, is called the range of the function. Usually the range of a function can be identified quite easily by inspecting its graph.

Example 1.2 Domain and Range of a Radical Function

Problem: For the function $h(x)$, determine its domain and range.

$$h(x) = \frac{1}{\sqrt{4-x^2}}$$

Solution:

Step 1: Find the domain.

The expression inside the square root must be strictly positive.

$$4 - x^2 > 0$$
$$\implies -2 < x < 2$$

$$\boxed{\text{Domain} : (-2, 2)}$$

Step 2: Find the range.

On the domain $(-2, 2)$, the value of the denominator is:

$$0 < \sqrt{4-x^2} \le 2$$

Taking the reciprocal of the inequality for $h(x)$ gives:

$$\frac{1}{\sqrt{4-x^2}} \ge \frac{1}{2}$$

The minimum value $(1/2)$ occurs at $x = 0$.

As $x \to \pm 2$, the denominator approaches 0^+, which means $h(x) \to \infty$.

$$\boxed{\text{Range} : \left[\frac{1}{2}, \infty\right)}$$

■

Q1. Find and sketch the domain and range of the following functions:

(a) $y = x^2 - 3$ **ans:** Domain $= (-\infty, \infty)$, Range $= [-3, \infty)$

(b) $y = \sqrt{x+3}$ **ans:** Domain $= [-3, \infty)$, Range $= [0, \infty)$

(c) $y = e^{x^2}$ **ans:** Domain $= (-\infty, \infty)$, Range $= [1, \infty)$

(d) $y = a\sin(bx)$ **ans:** Domain $= (-\infty, \infty)$, Range $= [-|a|, |a|]$

Q2. Consider the function given by $g(t) = 2t^2 + 1$ for $-2 \le t \le 2$.

(a) State the domain of the function. **ans:** Domain $= [-2, 2]$

(b) Plot a graph of the function.

(c) Deduce the range of the function from the graph. **ans:** Range $= [1, 9]$

Q3. Plot a graph of the following functions. In each case state the domain and the range of the function.

(a) $f(x) = 3x + 2$ for $-2 \le x \le 5$. **ans:** Domain $= [-2, 5]$, Range $= [-4, 17]$

(b) $g(x) = x^2 + 4$ for $-2 \le x \le 3$. **ans:** Domain $= [-2, 3]$, Range $= [4, 13]$

(c) $p(t) = 2t^2 + 8$ for $-2 \leq t \leq 4$. **ans:** Domain $= [-2, 4]$, Range $= [8, 40]$
(d) $f(t) = 6 - t^2$ for $1 \leq t \leq 5$. **ans:** Domain $= [1, 5]$, Range $= [-19, 5]$

1.3 Odd, even and periodic functions

An odd function is such that $f(-x) = -f(x)$ for all values of x. Any function which possesses such symmetry - that is the graph of the right can be obtained by rotating the curve on the left through 180° about the origin - is said to be an odd function.

An even function is such that $f(-x) = f(x)$ for all values of x. Any function which is symmetrical about the y axis, i.e. where the graph of the right-hand part is the mirror image of that on the left, is said to be an even function.

A function $f(x)$ is periodic if we can find a number T such that $f(x+T) = f(x)$ for all values of x. Any function that has a definite pattern repeated at regular intervals is said to be periodic. The interval over which the repetition takes place is called the period of the function, and is usually given the symbol T. The period of a periodic function is usually obvious from its graph.

Example 1.3 Testing for an Even Function

Problem: Given the function $g(x)$, show that it is an even function.

$$g(x) = x^4 - 6x^2 + 9$$

Solution:

Step 1: Replace every x with $-x$.

$$g(-x) = (-x)^4 - 6(-x)^2 + 9$$

Step 2: Simplify the powers.

$$g(-x) = x^4 - 6x^2 + 9$$

Step 3: Compare the result with the original function, $g(x)$.

$$g(-x) = g(x)$$

Since $g(-x) = g(x)$, the function $g(x)$ is even.

■

Example 1.4 Testing for an Odd Function

Problem: Given the function $f(x)$, show that it is an odd function.

$$f(x) = x^3 - 2x$$

Solution:

Step 1: Replace every x with $-x$.

$$f(-x) = (-x)^3 - 2(-x)$$

Step 2: Simplify the powers and signs.

$$f(-x) = -x^3 + 2x$$

Step 3: Factor out -1 to compare with the original function.

$$f(-x) = -(x^3 - 2x)$$

Step 4: Recognise that the expression in the parentheses is $f(x)$.

$$f(-x) = -f(x)$$

$$\boxed{\text{Since } f(-x) = -f(x), \text{ the function } f(x) \text{ is odd.}}$$

■

Example 1.5 Finding the Period of a Trigonometric Function

Problem: For the function $h(x)$, determine its period, T.

$$h(x) = \sin\left(3x - \frac{\pi}{4}\right)$$

Solution:

Step 1: Use the formula for the period of $\sin(kx + c)$.

The period is given by the formula $T = \dfrac{2\pi}{|k|}$.

For this function, $k = 3$, so we have: $T = \dfrac{2\pi}{3}$

Verification (from the definition $h(x+T) = h(x)$):

Step 2: Set up the equation $h(x+T) = h(x)$.

$$\sin\left(3(x+T) - \frac{\pi}{4}\right) = \sin\left(3x - \frac{\pi}{4}\right)$$

$$\sin\left(3x - \frac{\pi}{4} + 3T\right) = \sin\left(3x - \frac{\pi}{4}\right)$$

Step 3: Solve for the smallest positive value of T.

For the sine values to be equal, the term $3T$ must be a multiple of 2π.

$$3T = 2\pi n \quad \text{(for any integer } n\text{)}$$

The smallest positive period occurs when $n = 1$:

$$3T = 2\pi \implies T = \frac{2\pi}{3}$$

$$\boxed{\text{The period of } h(x) \text{ is } T = \frac{2\pi}{3}.}$$

■

Q1. Prove that $f(x) = x^3 + 4x$ is an odd function.

Q2. Prove that $f(x) = x^4 + 5$ is an even function.

Q3. Classify the following functions as odd, even or neither. If necessary sketch a graph to help you decide.

(a) $f(x) = x^2$ [**ans:** Even]
(b) $f(x) = 8x + 7$ [**ans:** Neither]
(c) $f(x) = 2x$ [**ans:** Odd]
(d) $f(x) = e^x$ [**ans:** Neither]

Q4. Determine whether the function is odd, even or periodic for:

(a) $f(x) = x^n$ [**ans:** Even if n is even, odd if n is odd]
(b) $f(x) = \dfrac{\cos(x)}{x^2}$ [**ans:** Even]
(c) $f(x) = \dfrac{\cosh(x)}{x}$ [**ans:** Odd]
(d) $f(x) = \dfrac{\tan(x)}{x}$ [**ans:** Even]
(e) $f(x) = x^{2n+1}$ [**ans:** Odd]

Q5. The function $f(x)$ satisfies

$$f(x) = \frac{3x+4}{x-2}, \quad x \in \mathbb{R},\ x \geq 3.$$

(a) Show that

$$f(x) = A + \frac{B}{x-2}$$

where constants A and B are positive integers to be found. [**ans:** A = 3, B = 10]
(b) Show that $f(x)$ is a decreasing function.
(c) Determine if the function $f(x)$ is odd, even, periodic or neither and explain your answer.

1.4 Inverse and composite functions

A composite function is a function that is formed by combining two functions, $f(x)$ and $g(x)$, such that the output of $g(x)$ becomes the input for $f(x)$. The composite function is denoted as: $f(g(x))$ where $g(x)$ is applied first, followed by $f(x)$. Note that the function $f(g(x))$ is different from $g(f(x))$.

An inverse function reverses the effect of a given function. If $f(x)$ is a function, its inverse $f^{-1}(x)$ satisfies the condition: $f(f^{-1}(x)) = x$,for all x in the domain of f^{-1} and f, respectively. The graph of $f(x)$ and $f^{-1}(x)$ is symmetric about the line $y = x$. Not all functions possess an inverse function. In fact, only one-to-one functions do so. If a function is many-to-one the process to reverse it would require many outputs from one input contradicting the definition of a function.

Example 1.6 Finding the Inverse of a Linear Function

Problem: Given the function $f(x)$, find its inverse, $f^{-1}(x)$.

$$f(x) = 3x - 4$$

Solution:

Step 1: Replace $f(x)$ with y.

$y = 3x - 4$

Step 2: Interchange the roles of x and y.

$x = 3y - 4$

Step 3: Solve the new equation for y.

$3y = x + 4$

$y = \frac{x+4}{3}$

Step 4: Replace y with $f^{-1}(x)$ to state the final answer.

The inverse function is $f^{-1}(x) = \frac{x+4}{3}$.

■

Example 1.7 Finding a Composite Function

Problem: Given the functions $f(x)$ and $h(x)$ find the composite function $f(h(x))$.

$f(x) = 2x + 3 \quad \text{and} \quad h(x) = x^2$

Solution:

Step 1: Substitute the inner function, $h(x)$, into the outer function, $f(x)$.
The notation $f(h(x))$ means we use the output of $h(x)$ as the input for $f(x)$.

$f(h(x)) = f(x^2)$

Step 2: Apply the rule for $f(x)$ to the new input, x^2.
The function f is defined as $f(\text{input}) = 2(\text{input}) + 3$.

$f(x^2) = 2(x^2) + 3$

The composite function is $f(h(x)) = 2x^2 + 3$.

■

Q1. Find $f(g(x))$ when $f(x) = x - 7$ and $g(x) = x^2$. [**ans:** $x^2 - 7$]

Q2. If $f(x) = x + 6$ and $g(x) = x^2 - 5$, find:

(a) $f(g(0))$ [**ans:** 1]
(b) $g(f(0))$ [**ans:** 31]
(c) $g(g(2))$ [**ans:** -4]
(d) $f(g(7))$ [**ans:** 50]

Q3. If $f(x) = \dfrac{x-3}{x+1}$ and $g(x) = \dfrac{3}{2x+1}$, find $g(f(x))$. $\left[\textbf{ans: } \dfrac{3(x+1)}{3x-5}\right]$

Q4. Find the inverse function of:

(a) $f(x) = 2x^3 + 7$ $\left[\textbf{ans: } f^{-1}(x) = \sqrt[3]{\dfrac{x-7}{2}}\right]$

(b) $f(x) = 4x + 2$ $\left[\textbf{ans: } f^{-1}(x) = \dfrac{x-2}{4}\right]$

Q5. Given the functions:

$$f(x) = e^{2x} - 4, \quad x \in \mathbb{R}.$$

$$g(x) = \frac{1}{x-11}, \quad x \in \mathbb{R},\ x \neq 11.$$

(a) Determine the range of $f(x)$. [**ans:** $f(x) > -4$]

(b) Find an expression for the inverse function $f^{-1}(x)$. $\left[\textbf{ans: } f^{-1}(x) = \dfrac{\ln(x+4)}{2}\right]$

(c) Find an expression for the inverse function $g^{-1}(x)$. $\left[\textbf{ans: } g^{-1}(x) = \dfrac{1}{x} + 11\right]$

Q6. The functions f and g are given by

$$f(x) = \sqrt{x}, \quad x \in \mathbb{R},\ x \geq 0,$$

$$g(x) = x - 2, \quad x \in \mathbb{R}.$$

(a) Find an expression for the function composition $fg(x)$. [**ans:** $\sqrt{x-2}$]

The function h, is defined by

$$h(x) = \sqrt{x-2}, \quad x \in \mathbb{R},\ 3 \leq x \leq 11.$$

(b) State the range of $h(x)$. [**ans:** $1 \leq h(x) \leq 3$]

(c) Determine an expression for the inverse function $h^{-1}(x)$. [**ans:** $h^{-1}(x) = x^2 + 2$]

(d) State the domain and range of $h^{-1}(x)$. **ans:** Domain: $[1,3]$, Range: $[3,11]$

Q7. The function f is defined as

$$f : x \mapsto \frac{2}{x-3} - \frac{4}{x^2-4x+3}, \quad x \in \mathbb{R},\ x > 1.$$

(a) Show clearly that

$$f : x \mapsto \frac{2}{x-1}, \quad x \in \mathbb{R},\ x > 1.$$

(b) Find an expression for $f^{-1}(x)$, in its simplest form. $\left[\textbf{ans: } f^{-1}(x) = \dfrac{2}{x} + 1\right]$

The function g is given by

$$g : x \mapsto 2x^2 + 4, \quad x \in \mathbb{R}.$$

(c) Solve the equation $f(g(x)) = \dfrac{4}{7}$. $\left[\textbf{ans: } x = \pm\dfrac{1}{2}\right]$

1.5 Chapter Review

Review Exercise 1 Mixed Questions

Problem 1.5.1 Given the function $f(x) = 5x + 18$, find:
(a) $f(5)$
(b) $f(9)$
(c) $f\left(-\frac{1}{4}\right)$

Problem 1.5.2 Given the functions $f(x) = \frac{x^2}{3}$ and $g(x) = 2x + 7$, find:
(a) $f(g(2))$
(b) $g(f(-3))$
(c) $f(f(4))$

Problem 1.5.3 Given the functions:

$$f(x) = \frac{1}{x^2 - 1}, \quad x \in \mathbb{R}, x > 1$$

$$g(x) = \frac{2}{x}, \quad x \in \mathbb{R}, x > 0$$

(a) Find the range of $f(x)$.
(b) Find the value of x for which $g(f(x)) = 6$.

Problem 1.5.4 Given the functions:

$$f(x) = \frac{x}{x - 2}, \quad x \in \mathbb{R}, x \neq 2$$

$$g(x) = \frac{3}{x}, \quad x \in \mathbb{R}, x \neq 0$$

(a) Find $f^{-1}(x)$.
(b) State the range of $f^{-1}(x)$.
(c) Find the value of x for which $f(g(x)) = 1.5$.

Problem 1.5.5 Consider a function $f : [0,1] \to [0,1]$ defined by $f(x) = x^2$. A new function $g(x)$ is constructed based on $f(x)$. Sketch $g(x)$ on the interval $[-4,4]$ given that:
(a) $g(x) = f(x)$ for $x \in [0,1]$, and $g(x)$ is periodic with period 1.
(b) $g(x) = f(x)$ for $x \in [0,1]$, $g(x)$ is odd, and is periodic with period 2.
(c) $g(x) = f(x)$ for $x \in [0,1]$, $g(x)$ is even, and is periodic with period 2.
(d) $g(x) = f(x)$ for $x \in [0,1]$, $g(x) = f(2-x)$ for $x \in [1,2]$, and is periodic with period 2.

Problem 1.5.6 Given the functions:

$$f(x) = e^x - 4, \quad x \in \mathbb{R}$$

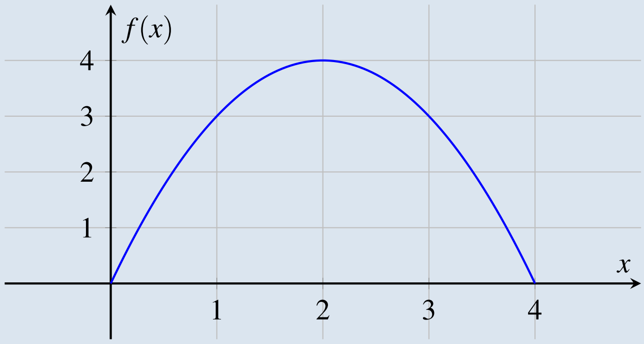
$$g(x) = \ln(x - 3), \quad x \in \mathbb{R}, x > 3$$

(a) Find the range of $f(x)$.
(b) Find the inverse function, $f^{-1}(x)$.
(c) Find the inverse function, $g^{-1}(x)$.

■

Review Exercise 2 Challenging Questions

Problem 1.5.7 The graph of the function $f(x) = -(x-2)^2 + 4$ is shown below.



(a) State the domain of $f(x)$.

(b) State the range of $f(x)$.

(c) Find the values of x for which $f(x) = 2$.

(d) Find the value of $f(1.5)$.

(e) State the interval(s) where $f(x)$ is increasing.

Problem 1.5.8 The function f is defined by

$$f(x) = \frac{4x-1}{2x+3}, \quad x \in \mathbb{R}, x \neq -\frac{3}{2}.$$

(a) Find the inverse function, $f^{-1}(x)$.

(b) Hence, solve the equation $f(x) = f^{-1}(x)$.

(c) The points where $f(x) = x$ would also satisfy the equation in part (b). Explain why there are no real solutions.

Problem 1.5.9 A function $f(x)$ is defined by the piecewise rule:

$$f(x) = \begin{cases} 3x-1, & x \leq 2 \\ x^2+1, & x > 2 \end{cases}$$

The function is one-to-one. Find the inverse function, $f^{-1}(x)$, stating its domain for each piece.

Problem 1.5.10 A function is given by the rule $f(x) = ax^2 + bx + c$. Find the values of the constants a, b, and c, given that the function satisfies all three of the following conditions:

- $f(x)$ is an even function.
- The graph of the function passes through the point $(2, 9)$.
- The range of the function is $[1, \infty)$.

■

Summary 1 Functions

Function: A rule that maps each input from a set (the domain) to a single, unique output in another set (the range).

Domain: The set of all valid inputs (x-values) for which the function is defined.

Range: The set of all possible outputs (y-values) produced by the function.

Even Function: A function that is symmetrical about the y-axis. The algebraic test is: $\mathbf{f(-x) = f(x)}$.

Odd Function: A function with 180° rotational symmetry about the origin. The algebraic test is: $\mathbf{f(-x) = -f(x)}$.

Periodic Function: A function that repeats its values at regular intervals. The smallest such interval is the period, T. The algebraic test is: $\mathbf{f(x+T) = f(x)}$.

Composite Function: A function created by applying one function to the result of another. The notation is: $\mathbf{(f \circ g)(x) = f(g(x))}$.

Inverse Function: A function, denoted $f^{-1}(x)$, that reverses the action of a one-to-one function $f(x)$. The domain of $f^{-1}(x)$ is the range of $f(x)$. The graph of $f^{-1}(x)$ is a reflection of $f(x)$ in the line $y = x$.

Chapter Checklist

- ☐ I can define the terms domain and range, and determine them for a given function.
- ☐ I can evaluate a function for a given numerical or algebraic input.
- ☐ I can test if a function is odd, even, or periodic using the algebraic definitions.
- ☐ I can find the composite function $f(g(x))$ and state its domain.
- ☐ I can find the inverse of a one-to-one function, $f^{-1}(x)$.
- ☐ I understand the relationship between the graph, domain, and range of a function and its inverse.

■

2. Differentiation

2.1 Differentiation of standard functions

Differentiation is a mathematical technique which can be used for analysing the way in which functions change. In particular, it measures the rate of change of a function at any given point. In essence finding the gradient.

In engineering applications the function may, for example, model and predict changes in velocity, altitude, and trajectory, model the rate of temperature changes over time or space in thermal systems, analysis of changes in heart rate or blood pressure over time, determine torque and changes in wind speed affecting performance and so much more.

In this chapter we explain what is meant by the gradient of a curve and introduce differentiation as a method for finding the gradient at any point.

Function	**Derivative**
Basic Rules	
constant	0
x^n	nx^{n-1}
kx^n	knx^{n-1}
Exponential & Logarithmic	
e^x	e^x
e^{kx}	ke^{kx}
$\ln x$	$\frac{1}{x}$

Function	**Derivative**
Trigonometric Functions	
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
Trigonometric (Chain Rule)	
$\sin(kx+c)$	$k\cos(kx+c)$
$\cos(kx+c)$	$-k\sin(kx+c)$
$\tan(kx+c)$	$k\sec^2(kx+c)$

2.1.1 Polynomial functions

Example 2.1 Differentiating a Polynomial

Problem: Differentiate the polynomial function:

$$f(x) = 4x^5 - 3x^3 + 2x^2 - 7x + 5$$

Solution:

Step 1: Apply the power rule, $\frac{d}{dx}(x^n) = nx^{n-1}$, to each term of the polynomial.

$$f'(x) = \frac{d}{dx}(4x^5) - \frac{d}{dx}(3x^3) + \frac{d}{dx}(2x^2) - \frac{d}{dx}(7x) + \frac{d}{dx}(5)$$

Step 2: Calculate the derivative of each term individually.

$$f'(x) = (4 \cdot 5x^4) - (3 \cdot 3x^2) + (2 \cdot 2x^1) - (7) + (0)$$

Step 3: Simplify the resulting expression.

$$f'(x) = 20x^4 - 9x^2 + 4x - 7$$

$$\boxed{f'(x) = 20x^4 - 9x^2 + 4x - 7}$$

■

Q1. Find the derivative of $y = 7x + \sqrt{x}$. $\left[\textbf{ans: } \frac{dy}{dx} = 7 + \frac{1}{2\sqrt{x}}\right]$

Q2. Find the derivative of $y = x^3 + 7$. $\left[\textbf{ans: } \frac{dy}{dx} = 3x^2\right]$

Q3. Find the derivative of $y = 2x^2$. $\left[\textbf{ans: } \frac{dy}{dx} = 4x\right]$

Q4. Find the derivative of $y = 1 - 6x^{\frac{5}{2}}$. $\left[\textbf{ans: } \frac{dy}{dx} = -15x^{\frac{3}{2}}\right]$

Q5. Find the derivative of $y = 5x^4 + 2x^3 + 4x + 7$. $\left[\textbf{ans: } \frac{dy}{dx} = 20x^3 + 6x^2 + 4\right]$

Q6. Find the derivative of $y = \frac{3}{2x^2} + \frac{7}{3x^3} + 18x + 27$. $\left[\textbf{ans: } \frac{dy}{dx} = -\frac{3}{x^3} - \frac{7}{x^4} + 18\right]$

Q7. Find the derivative of $x = \frac{3}{2}t$ with respect to t. $\left[\textbf{ans: } \frac{dx}{dt} = \frac{3}{2}\right]$

Q8. Find the derivative of $x = 113$ with respect to t. $\left[\textbf{ans: } \frac{dx}{dt} = 0\right]$

Q9. Find the derivative of $A = \pi r^2$ with respect to r. $\left[\textbf{ans: } \frac{dA}{dr} = 2\pi r\right]$

Q10. For $v = u + at$, find $\frac{dv}{dt}$, assuming u and a are constants. $\left[\textbf{ans: } \frac{dv}{dt} = a\right]$

2.1.2 Trigonometric

Example 2.2 Differentiating Trigonometric Functions

Problem: Differentiate the function:

$$f(x) = \sin x + 3\cos x - 2\tan x$$

Solution:

Step 1: Recall the standard derivatives for trigonometric functions.

$$\frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x, \quad \frac{d}{dx}(\tan x) = \sec^2 x$$

Step 2: Apply these rules to differentiate the function term-by-term.

$$f'(x) = \frac{d}{dx}(\sin x) + 3\frac{d}{dx}(\cos x) - 2\frac{d}{dx}(\tan x)$$

Step 3: Substitute the derivatives and simplify.

$$f'(x) = (\cos x) + 3(-\sin x) - 2(\sec^2 x)$$
$$f'(x) = \cos x - 3\sin x - 2\sec^2 x$$

$$\boxed{f'(x) = \cos x - 3\sin x - 2\sec^2 x}$$

■

Q1. Find the derivative of $y = \sin x$. $\left[\textbf{ans: } \frac{dy}{dx} = \cos x\right]$

Q2. Find the derivative of $y = 3\cos x + 2\sin x$. $\left[\textbf{ans: } \frac{dy}{dx} = -3\sin x + 2\cos x\right]$

Q3. Find the derivative of $y = x^3 \sin 3x$. $\left[\textbf{ans: } \frac{dy}{dx} = 3x^2 \sin 3x + 3x^3 \cos 3x\right]$

Q4. Find the derivative of $y = \frac{3}{4}\tan(4x+1)$. $\left[\textbf{ans: } \frac{dy}{dx} = 3\sec^2(4x+1)\right]$

Q5. Find the derivative of $y = \frac{3}{2}\cos 4x - \sin(3x+2)$. $\left[\textbf{ans: } \frac{dy}{dx} = -6\sin 4x - 3\cos(3x+2)\right]$

Q6. Find the derivative of $y = (x^2+3x+2)\cos 2x$.

$$\left[\textbf{ans: } \frac{dy}{dx} = (2x+3)\cos 2x - 2(x^2+3x+2)\sin 2x\right]$$

Q7. Find the derivative of $y = \cos^4 x$. $\left[\textbf{ans: } \frac{dy}{dx} = -4\cos^3 x \sin x\right]$

Q8. Find the derivative of $y = \frac{x}{2-\tan x}$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{2-\tan x + x\sec^2 x}{(2-\tan x)^2}\right]$

Q9. Find the derivative of $y = 2\arctan\sqrt{\sin x}$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{\cos x}{(1+\sin x)\sqrt{\sin x}}\right]$

Q10. Find the derivative of $y = (\sin x)^{\cos x}$. $\left[\textbf{ans: } \frac{dy}{dx} = (\sin x)^{\cos x}(\cos x \cot x - \sin x \ln(\sin x))\right]$

2.1.3 Exponential

Example 2.3 Differentiating Exponential Functions

Problem: Differentiate the function:

$$f(x) = 5e^x + 3e^{2x} - 4e^{-x}$$

Solution:

Step 1: Recall the derivative rule for exponential functions, $\frac{d}{dx}(e^{kx}) = ke^{kx}$.

Step 2: Apply this rule to differentiate the function term-by-term.

$$f'(x) = \frac{d}{dx}(5e^x) + \frac{d}{dx}(3e^{2x}) - \frac{d}{dx}(4e^{-x})$$

Step 3: Calculate the derivative of each term and simplify.

$$f'(x) = (5e^x) + (3\cdot 2e^{2x}) - (4\cdot(-1)e^{-x})$$
$$f'(x) = 5e^x + 6e^{2x} + 4e^{-x}$$

$$\boxed{f'(x) = 5e^x + 6e^{2x} + 4e^{-x}}$$

Q1. Find the derivative of $y = e^{2x} - e^x$. $\left[\textbf{ans: } \frac{dy}{dx} = 2e^{2x} - e^x\right]$

Q2. Find the derivative of $y = 7e^{3x}$. $\left[\textbf{ans: } \frac{dy}{dx} = 21e^{3x}\right]$

Q3. Find the derivative of $y = 23e^{2x} + 15e^{3x}$. $\left[\textbf{ans: } \frac{dy}{dx} = 46e^{2x} + 45e^{3x}\right]$

Q4. Find the derivative of $y = \frac{3}{2e^{2x}} + \frac{7}{3e^{3x}} + 17e^x + 15e^2$. $\left[\textbf{ans: } \frac{dy}{dx} = -3e^{-2x} - 7e^{-3x} + 17e^x\right]$

Q5. Find the derivative of $y = e^{\sqrt{x}}$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}\right]$

2.1.4 Logarithmic

Example 2.4 Differentiating Logarithmic Functions (Chain Rule)

Problem: Differentiate the function:

$$f(x) = \ln(x^2 + 3x)$$

Solution:

Step 1: Recall the chain rule for logarithmic functions.
The rule is $\frac{d}{dx}[\ln(u)] = \frac{1}{u} \cdot \frac{du}{dx}$.
For this problem, we identify the inner function as $u = x^2 + 3x$.

Step 2: Differentiate the inner function, u, with respect to x.

$$\frac{du}{dx} = \frac{d}{dx}(x^2 + 3x) = 2x + 3$$

Step 3: Substitute u and $\frac{du}{dx}$ back into the chain rule formula.

$$f'(x) = \frac{1}{u} \cdot \frac{du}{dx} = \frac{1}{x^2 + 3x} \cdot (2x + 3)$$

Step 4: Simplify the resulting expression.

$$f'(x) = \frac{2x + 3}{x^2 + 3x}$$

$$\boxed{f'(x) = \frac{2x + 3}{x^2 + 3x}}$$

Q1. Find the derivative of $y = 5\ln x$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{5}{x}\right]$

Q2. Find the derivative of $y = \ln 2x$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{1}{x}\right]$

Q3. Find the derivative of $y = 3\ln x + 2\ln 3x$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{5}{x}\right]$

Q4. Find the derivative of $y = \frac{3\ln 2x}{2} + \frac{7\ln 5x}{3} + 14\ln 2$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{23}{6x}\right]$

Q5. Find the derivative of $y = \ln|x^2 - 1|$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{2x}{x^2 - 1}\right]$

Q6. Find the derivative of $y = \ln\left(\frac{x+1}{x-1}\right)$. $\left[\textbf{ans: } \frac{dy}{dx} = -\frac{2}{x^2 - 1}\right]$

Q7. Find the derivative of $y = x^{\frac{1}{2}}\ln(3x)$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{\ln(3x)+2}{2\sqrt{x}}\right]$

Q8. Find the derivative of $y = \sqrt{\ln x}$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{1}{2x\sqrt{\ln x}}\right]$

Q9. Find the derivative of $y = \frac{1}{1+\ln x}$. $\left[\textbf{ans: } \frac{dy}{dx} = -\frac{1}{x(1+\ln x)^2}\right]$

Q10. Find the derivative of $y = \ln\left(\frac{(4-8x)^2}{(2-4x)^4}\right)$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{4}{1-2x}\right]$

2.2 Differentiation techniques

The first two sections covered differentiating from first principles and the differentiation of standard functions such as $x^2, e^x, \cos(2x)$ and $\ln(x)$.

Now in this section slightly more complex functions will be differentiated such as $e^{ax}x^n, \frac{\tan bx+c}{e^a x}$. In order to differentiate these functions various new methods will be required which you should have already covered in A-level maths. In this section we will cover Chain rule, Product rule and Quotient rule.

2.2.1 The Chain Rule

The Chain rule is the differentiation technique used to differentiate composite functions which we covered in the previous chapter, or in simpler terms a function within another function.

If a function y is a composition of two functions, written as $y = f(g(x))$, we can define an intermediate variable. Let $u = g(x)$, so that $y = f(u)$. The derivative of y with respect to x is then given by:

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

Alternatively, using prime notation, the rule can be expressed in a single step as:

$$\frac{dy}{dx} = f'(g(x))\cdot g'(x)$$

This means you take the derivative of the **outer function** (leaving the inner function unchanged) and multiply it by the derivative of the **inner function**.

Example 2.5 Applying the Chain Rule

Problem: Differentiate $y = \sin(x^2+1)$ with respect to x.

Solution:

Step 1: Identify the inner and outer functions.

Let the inner function be $u = x^2 + 1$.

This makes the outer function $y = \sin(u)$.

Step 2: Differentiate both functions with respect to their variables.

$$\frac{du}{dx} = 2x$$
$$\frac{dy}{du} = \cos(u)$$

Step 3: Apply the chain rule formula, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

$$\frac{dy}{dx} = \cos(u) \cdot (2x)$$

Step 4: Substitute the expression for u back into the equation and simplify.

$$\frac{dy}{dx} = \cos(x^2 + 1) \cdot (2x)$$
$$\frac{dy}{dx} = 2x\cos(x^2 + 1)$$

$$\boxed{\frac{dy}{dx} = 2x\cos(x^2 + 1)}$$

■

Q1. Find the derivative of $y = (x+1)^6$. $\left[\textbf{ans: } \frac{dy}{dx} = 6(x+1)^5\right]$

Q2. Find the derivative of $y = e^{7x^2+3x+4}$. $\left[\textbf{ans: } \frac{dy}{dx} = (14x+3)e^{7x^2+3x+4}\right]$

Q3. Find the derivative of $y = \cos^2 x$. $\left[\textbf{ans: } \frac{dy}{dx} = -2\sin x\cos x\right]$

Q4. Find the derivative of $y = \sin(x^2 + 2x)$. $\left[\textbf{ans: } \frac{dy}{dx} = 2(x+1)\cos(x^2+2x)\right]$

Q5. Find the derivative of $y = \frac{5}{(2x+3)^3}$. $\left[\textbf{ans: } \frac{dy}{dx} = -\frac{30}{(2x+3)^4}\right]$

Q6. Find the derivative of $y = \frac{1}{\sqrt{2\cos x + 5}}$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{\sin x}{(2\cos x+5)^{3/2}}\right]$

Q7. Find the derivative of $v = \sqrt{2 + \ln 3t}$ with respect to t. $\left[\textbf{ans: } \frac{dv}{dt} = \frac{1}{2t\sqrt{2+\ln(3t)}}\right]$

Q8. Find the derivative of $S = 2^r$ with respect to r. $\left[\textbf{ans: } \frac{dS}{dr} = 2^r \ln 2\right]$

2.2.2 The Product Rule

The product rule is used to find the derivative of a product of two functions.
If a function y is the product of two differentiable functions, say $g(x)$ and $h(x)$, so that $y = g(x)h(x)$, then its derivative is given in prime notation as:

$$y' = g'(x)h(x) + g(x)h'(x)$$

Alternatively, using Leibniz notation, we can let $u = g(x)$ and $v = h(x)$, so that $y = uv$. The rule is then expressed as:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Example 2.6 Applying the Product Rule

Problem: Differentiate $y = x^2 \ln(x)$ with respect to x.

Solution:

Step 1: Identify the two functions being multiplied, u and v.

$$u = x^2$$
$$v = \ln x$$

Step 2: Differentiate u and v with respect to x.

$$\frac{du}{dx} = 2x$$
$$\frac{dv}{dx} = \frac{1}{x}$$

Step 3: Apply the product rule formula, $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

$$\frac{dy}{dx} = (x^2)\left(\frac{1}{x}\right) + (\ln x)(2x)$$

Step 4: Simplify the resulting expression.

$$\frac{dy}{dx} = x + 2x\ln x$$
$$\frac{dy}{dx} = x(1 + 2\ln x)$$

$$\boxed{\frac{dy}{dx} = x(1 + 2\ln x)}$$

■

Q1. Find the derivative of $y = 7x\cos x$. $\left[\textbf{ans: } \frac{dy}{dx} = 7\cos x - 7x\sin x\right]$

Q2. Find the derivative of $y = 5x^4e^{7x}$. $\left[\textbf{ans: } \frac{dy}{dx} = 5x^3e^{7x}(4 + 7x)\right]$

Q3. Find the derivative of $y = \sin x \cos^2 x$. $\left[\textbf{ans: } \frac{dy}{dx} = \cos x(\cos^2 x - 2\sin^2 x)\right]$

Q4. Find the derivative of $y = e^{2x}(4\sin 2x + 3\cos 2x)$. $\left[\textbf{ans: } \frac{dy}{dx} = 2e^{2x}(\sin 2x + 7\cos 2x)\right]$

Q5. Find the derivative of $y = \cos 2x + \tan x \sin 2x$.
$\left[\textbf{ans: } \frac{dy}{dx} = -2\sin 2x + \sec^2 x \sin 2x + 2\tan x \cos 2x\right]$

Q6. Find the derivative of $v = 6\sqrt{t}(2t-1)^4$ with respect to t. $\left[\textbf{ans: } \frac{dv}{dt} = \frac{3(2t-1)^3}{\sqrt{t}}(18t-1)\right]$

Q7. The curve C has equation $y = x\ln x$, for $x > 0$. Find the exact coordinates of the turning point of C. $\left[\textbf{ans: } \left(\frac{1}{e}, -\frac{1}{e}\right)\right]$

Q8. A curve C has equation $y = xe^{2x}$, for $x \in \mathbb{R}$. Show that an equation of the tangent to C at the point where $x = \frac{1}{2}$ is $2y = e(4x-1)$

Q9.

(a) Find $\frac{d}{dx}\left(x^2 \cot 2x\right)$. $[\textbf{ans: } 2x\cot 2x - 2x^2 \csc^2 2x]$

(b) Show that $\frac{d}{dx}(\tan x) = \sec^2 x$.

Q10. The curve C has equation $y = (x-1)(x-2) + \ln x$, for $x > 0$.

(a) Show that one of the turning points of C has coordinates $\left(\frac{1}{2}, \frac{3}{4} - \ln 2\right)$ and find the coordinates of the other. [**ans:** (1,0)]

(b) Determine the nature of the turning point with coordinates $\left(\frac{1}{2}, \frac{3}{4} - \ln 2\right)$.
[**ans:** local maximum]

Q11. The curve C has equation

$$y = \frac{kx^2 - a}{kx^2 + a},$$

where k and a are non-zero constants.

(a) Find a simplified expression for $\frac{dy}{dx}$ in terms of a and k. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{4akx}{(kx^2+a)^2}\right]$

(b) Hence show that C has a single turning point for all values of a and k, and state its coordinates. [**ans:** (0, -1)]

Q13. The curve C has equation

$$y = \frac{x^2 - 6x + 12}{4x - 11}, \quad x \in \mathbb{R},\ x \neq \frac{11}{4}.$$

(a) Find a simplified expression for $\frac{dy}{dx}$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{2(2x-9)(x-1)}{(4x-11)^2}\right]$

(b) Determine the range of values of x for which y is decreasing. [**ans:** $1 < x < \frac{9}{2}, x \neq \frac{11}{4}$]

2.2.3 The Quotient Rule

The quotient rule is used to find the derivative of a function that is a quotient of two other functions.If a function y can be written as $y = \frac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are differentiable functions, its derivative in prime notation is:

$$y' = \frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}$$

Alternatively, using Leibniz notation, we can let $u = g(x)$ and $v = h(x)$, so that $y = \frac{u}{v}$. The rule is then expressed as:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

The order of the terms in the numerator is very important due to the subtraction.

Example 2.7 Applying the Quotient Rule

Problem: Differentiate $y = \frac{x^2}{\ln x}$ with respect to x.
Solution:

Step 1: Identify the numerator, u, and the denominator, v.

$$u = x^2$$
$$v = \ln x$$

Step 2: Differentiate u and v with respect to x.

$$\frac{du}{dx} = 2x$$
$$\frac{dv}{dx} = \frac{1}{x}$$

Step 3: Apply the quotient rule formula, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

$$\frac{dy}{dx} = \frac{(\ln x)(2x) - (x^2)\left(\frac{1}{x}\right)}{(\ln x)^2}$$

Step 4: Simplify the resulting expression.

$$\frac{dy}{dx} = \frac{2x\ln x - x}{(\ln x)^2}$$
$$\frac{dy}{dx} = \frac{x(2\ln x - 1)}{(\ln x)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{x(2\ln x - 1)}{(\ln x)^2}}$$

■

Q1. Find the derivative of $y = \frac{(x+1)^6}{(2x+8)^5}$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{2(x+19)(x+1)^5}{(2x+8)^6}\right]$

Q2. Find the derivative of $y = \frac{x(2x^2+2)}{(2x+1)^2}$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{2(2x^3+3x^2-2x+1)}{(2x+1)^3}\right]$

Q3. Find the derivative of $y = \frac{2x+1}{\sqrt{x+1}}$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{2x+3}{2(x+1)^{3/2}}\right]$

Q4. Find the derivative of $y = \frac{\sec x}{\sin x}$. $\left[\textbf{ans: } \frac{dy}{dx} = -\frac{\cos(2x)}{\sin^2 x \cos^2 x}\right]$

Q5. Find the derivative of $y = \frac{\ln x}{\sqrt{x}}$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{2-\ln x}{2x^{3/2}}\right]$

Q6. Find the derivative of $y = \frac{4e^x}{e^x+2}$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{8e^x}{(e^x+2)^2}\right]$

Q7. Find the derivative of $y = \frac{\sqrt{e^{2x}-9}}{e^x}$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{9}{e^x\sqrt{e^{2x}-9}}\right]$

Q8. Find the derivative of $y = \frac{2x(x^2+6x+12)}{(x+2)^3}$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{48}{(x+2)^4}\right]$

Q9. Find the derivative of $y = \sqrt{\frac{x+1}{x-1}}$. $\left[\textbf{ans: } \frac{dy}{dx} = -\frac{1}{\sqrt{x+1}(x-1)^{3/2}}\right]$

Q10. Find the derivative of $y = \frac{\cos 2x}{\sqrt{1+\sin 2x}}$. $\left[\textbf{ans: } \frac{dy}{dx} = -\sqrt{1+\sin 2x}\right]$

2.3 Implicit Differentiation

When a relationship between x and y is defined by an equation that is difficult to solve for y (e.g., $x^3+y^3=6xy$), we can find the derivative using implicit differentiation. The process involves differentiating both sides of the equation with respect to x, while treating y as an unknown function of x. The key is to apply the chain rule whenever differentiating a term containing y. The general form of this rule is:

$$\frac{d}{dx}f(y) = f'(y)\cdot\frac{dy}{dx}$$

For example, when differentiating a power of y, this application of the chain rule gives:

$$\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$$

Similarly, when differentiating a product like xy, we apply the standard product rule:

$$\frac{d}{dx}(xy) = (1\cdot y) + \left(x\cdot\frac{dy}{dx}\right) = y + x\frac{dy}{dx}$$

After differentiating all terms, you rearrange the equation to solve for $\frac{dy}{dx}$. The final expression will often be in terms of both x and y.

Example 2.8 Applying Implicit Differentiation

Problem: Find $\frac{dy}{dx}$ for the curve defined by the equation:

$$x^2 + y^2 = 25$$

Solution:

Step 1: Differentiate both sides of the equation with respect to x.

We apply the chain rule to the term involving y, since y is a function of x.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

Step 2: Rearrange the equation to solve for $\frac{dy}{dx}$.

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

Step 3: Simplify the final expression.

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

■

Q1. For the curve $x^2 + 2xy + 3y^2 = 12$, find $\frac{dy}{dx}$. $\left[\textbf{ans: } \frac{dy}{dx} = -\frac{x+y}{x+3y}\right]$

Q2. For $y^3 - x^2y^2 = x^2 + 3x + 1$, find $\frac{dy}{dx}$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{2x+3+2xy^2}{y(3y-2x^2)}\right]$

Q3. For $(3x-y)(2x+3y) = 8$, find $\frac{dy}{dx}$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{12x+7y}{6y-7x}\right]$

Q4. For $y(x^3+y^3) = (x+1)(x+4)$, find $\frac{dy}{dx}$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{2x+5-3x^2y}{x^3+4y^3}\right]$

Q5. For $ye^x = xe^y$, find $\frac{dy}{dx}$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{e^y - ye^x}{e^x - xe^y}\right]$

Q6. For $\frac{x^2}{x+2y} = 3y^2$, find $\frac{dy}{dx}$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{2x-3y^2}{6y(x+3y)}\right]$

Q7. For $3x^2 + 3 = \ln(5xy^2)$, find $\frac{dy}{dx}$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{y(6x^2-1)}{2x}\right]$

Q8. For $\sin y = -e^y + x^3 + 2$, find $\dfrac{dy}{dx}$. $\left[\textbf{ans: } \dfrac{dy}{dx} = \dfrac{3x^2}{\cos y + e^y}\right]$

Q9. For $x^2 + y^2 = 16$, find $\dfrac{dy}{dx}$. $\left[\textbf{ans: } \dfrac{dy}{dx} = -\dfrac{x}{y}\right]$

Q10. For the curve $\dfrac{(x+2y)^2}{4x-y} + y = x$, find $\dfrac{dy}{dx}$. $\left[\textbf{ans: } \dfrac{dy}{dx} = \dfrac{2x-3y}{3x+2y}\right]$

Q11. Find the equation of the tangent to the curve $y^2 + 3xy - 2x^2 + 17 = 0$ at the point (2,3).
[**ans:** $x + 12y = 38$]

Q12. For the curve $2y^2 - xy + 4x + x^2 = 7$:

(a) Find an expression for $\dfrac{dy}{dx}$. $\left[\textbf{ans: } \dfrac{dy}{dx} = \dfrac{y-2x-4}{4y-x}\right]$
(b) Find the coordinates of the two stationary points. $\left[\textbf{ans: } (-1,2) \text{ and } \left(-\frac{25}{7}, -\frac{22}{7}\right)\right]$

Q13. For the curve $4xy - (x+2)^2 = y^2 - 5$:

(a) Find an expression for $\dfrac{dy}{dx}$. $\left[\textbf{ans: } \dfrac{dy}{dx} = \dfrac{2y-x-2}{y-2x}\right]$
(b) Determine the coordinates of the two stationary points. [**ans:** (0,1) and $\left(\frac{4}{3}, \frac{5}{3}\right)$]

Q14. The normal to the curve $ax^3 - 3xy + by^2 = 224$ at the point $P(-2,6)$ has the equation $15x - 13y + 108 = 0$. Find the values of the constants a and b.
[**ans:** $a = 8, b = 7$]

Q15. For the curve defined by $y(xy+3) = 2x+1$:

(a) Find an expression for $\dfrac{dy}{dx}$. $\left[\textbf{ans: } \dfrac{dy}{dx} = \dfrac{2-y^2}{2xy+3}\right]$
(b) Show that there is no point on the curve where the tangent is parallel to the y-axis.

Q16. A curve has the implicit equation $8x^4 + 32xy^3 + 16y^4 = 1$. Find the coordinates of any real points on the curve where the gradient is $\frac{1}{2}$.
[**ans:** No Real Points]

Q17. A curve C has the implicit equation $(xy-2)(y+5) = 10$. The curve crosses the y-axis at point A, and line L is the tangent to C at A.

(a) State the coordinates of A. [**ans:** (0, -10)]
(b) Find an equation for L. [**ans:** $y = 25x - 10$]
(c) Find the coordinates of the point where L meets C again. [**ans:** $\left(\frac{3}{5}, 5\right)$]

Q18. For the curve $4^x + 2xy + y^2 = 13$:

(a) Show that $\dfrac{dy}{dx} = -\dfrac{y + 4^x \ln 2}{x+y}$.
(b) Find the gradient at the two points on the curve where $x = 2$.
[**ans:** $1 - 16\ln 2$ and $-3 + 16\ln 2$]

Q19. Show that for the curve $ye^y = x^x$, where $x > 0$, the derivative is $\dfrac{dy}{dx} = \dfrac{y(1+\ln x)}{1+y}$.

Q20. For the curve $4y^2 + 3xy - 2x^2 = 2x - 2y - 12$:

(a) Show that $\dfrac{dy}{dx} = \dfrac{2+4x-3y}{8y+3x+2}$.
(b) Find the coordinates of the two points where the tangent to the curve has a gradient of -2.

2.4 Parametric Differentiation

When x and y are both defined as functions of a third variable, known as a parameter (commonly t), we can find the derivative $\frac{dy}{dx}$ without first converting to a Cartesian equation. Given $x = x(t)$ and $y = y(t)$, we can use the chain rule, which states:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

By rearranging this equation, we can solve for $\frac{dy}{dx}$ to get the formula for parametric differentiation, provided that $\frac{dx}{dt} \neq 0$:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Example 2.9 Applying Parametric Differentiation

Problem: Find $\dfrac{dy}{dx}$ for the curve defined by the parametric equations:

$$x = t^2 + 1, \quad y = t^3 + 2t$$

Solution:

Step 1: Differentiate both x and y with respect to the parameter t.

$$\frac{dx}{dt} = \frac{d}{dt}(t^2+1) = 2t$$
$$\frac{dy}{dt} = \frac{d}{dt}(t^3+2t) = 3t^2+2$$

Step 2: Apply the formula for parametric differentiation, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$.

$$\frac{dy}{dx} = \frac{3t^2+2}{2t}$$

$$\boxed{\frac{dy}{dx} = \frac{3t^2+2}{2t}}$$

■

Q1. A curve is given parametrically by $x = 3 + 2\cos\theta$ and $y = -3 + 2\sin\theta$, for $0 \leq \theta < 2\pi$. Show clearly that $\dfrac{dy}{dx} = \dfrac{3-x}{y+3}$.

Q2. A curve is defined by the parametric equations $x = \frac{1}{2}a\cos\theta$ and $y = a\sin\theta$, where a is a positive constant. Show clearly that $\dfrac{dy}{dx} = -\dfrac{4x}{y}$.

Q3. A curve is given by $x = 1 - \cos 2\theta$ and $y = \sin 2\theta$. The point P lies on this curve where $\theta = \frac{\pi}{6}$. Show that an equation of the normal to the curve at P is $y + \sqrt{3}x = \sqrt{3}$.

Q4. A curve is defined by the parametric equations $x = t + 1$ and $y = t^2 - 1$. Find the Cartesian equation of the curve.

[**ans:** $y = x^2 - 2x$]

Q5. A curve is defined by $x = a\cos\theta$ and $y = a\sin^2\theta$, where a is a positive constant. Show that the equation of the tangent to the curve where $\theta = \frac{\pi}{3}$ is $4x + 4y = 5a$.

Q6. A curve C is given by $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$. Find the coordinates of the points of intersection between C and the line $3y = 4x$.

[**ans:** $\left(\frac{3}{5}, \frac{4}{5}\right)$ and $\left(-\frac{3}{5}, -\frac{4}{5}\right)$]

Q7. A curve is given by $x = \frac{a}{2}(2t - \sin 2t)$ and $y = a(1 - \cos 2t)$. Find the equation of the tangent to the curve where $t = \frac{\pi}{4}$.
[**note:** The target equation in the original problem was incorrect.]

[**ans:** $y - a = 2\left(x - \frac{a(\pi-2)}{4}\right)$]

Q8. A curve is given by $x = \frac{2}{t}$ and $y = t^2 - 1$. The point $P(4, y)$ lies on this curve. Show that an equation of the tangent to the curve at P is $x + 8y + 2 = 0$.

Q9. For the curve defined by $x = 2\cos t$ and $y = \sqrt{3}\cos 2t$, find an expression for $\dfrac{dy}{dx}$.

$\left[\textbf{ans: } \dfrac{dy}{dx} = 2\sqrt{3}\cos t\right]$

Q10. For the curve defined by $x = 4\theta - \cos\theta$ and $y = 1 + \sin\theta$, find an expression for $\dfrac{dy}{dx}$.

$\left[\textbf{ans: } \dfrac{dy}{dx} = \dfrac{\cos\theta}{4+\sin\theta}\right]$

2.5 Stationary Points of Single-Variable Functions

Differentiation can be used to find the stationary points of a function's graph. A stationary point is any point on a curve where the gradient is zero, i.e., where: $\frac{dy}{dx} = 0$.

These points are candidates for turning points (local maxima or minima), but can also be stationary points of inflection. For example, the point $(0,0)$ on the curve $y = x^3$ is a stationary point but not a turning point. Thus, while all turning points are stationary points, not all stationary points are turning points.

The First Derivative

The sign of the first derivative, $\frac{dy}{dx}$, tells us whether the function is increasing or decreasing:

- **Increasing Function:** The function is increasing where $\frac{dy}{dx} > 0$.
- **Decreasing Function:** The function is decreasing where $\frac{dy}{dx} < 0$.

The Second Derivative

The second derivative, $\frac{d^2y}{dx^2}$, provides information about the **concavity** (how the curve bends) and can be used to classify stationary points.

- **Concave (Concave Down):** The curve is concave where $\frac{d^2y}{dx^2} < 0$.
- **Convex (Concave Up):** The curve is convex where $\frac{d^2y}{dx^2} > 0$.
- **Point of Inflection:** This is a point where the concavity changes. A candidate point exists if $\frac{d^2y}{dx^2} = 0$, but you must confirm that the sign of $\frac{d^2y}{dx^2}$ changes on either side of the point.

- **Local Maximum:** The point is a local maximum if $\frac{d^2y}{dx^2} < 0$.
- **Local Minimum:** The point is a local minimum if $\frac{d^2y}{dx^2} > 0$.
- **Inconclusive:** If $\frac{d^2y}{dx^2} = 0$, this test fails. You must use the first derivative test (checking the sign of the gradient on either side of the point) to classify it.

Example 2.10 Finding and Classifying Stationary Points

Problem: Find and classify the stationary points of the function:

$$f(x) = x^3 - 6x^2 + 9x + 1$$

Solution:

Step 1: Find the first derivative, $f'(x)$.

$f'(x) = 3x^2 - 12x + 9$

Step 2: Set $f'(x) = 0$ to find the x-coordinates of the stationary points.

$3x^2 - 12x + 9 = 0$

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$

$\implies x = 1$ and $x = 3$

Step 3: Find the second derivative, $f''(x)$, to classify the points.

$f''(x) = 6x - 12$

Step 4: Apply the second derivative test to each x-value.

At $x = 1$: $f''(1) = 6(1) - 12 = -6$. (Since $f''(1) < 0 \implies$ Local Maximum)

At $x = 3$: $f''(3) = 6(3) - 12 = 6$. (Since $f''(3) > 0 \implies$ Local Minimum)

Step 5: Find the y-coordinates to determine the full coordinate points.

At $x = 1$: $f(1) = (1)^3 - 6(1)^2 + 9(1) + 1 = 5$.

At $x = 3$: $f(3) = (3)^3 - 6(3)^2 + 9(3) + 1 = 1$.

Local Maximum at $(1,5)$
Local Minimum at $(3,1)$

■

Q1. The curve C has equation $y = \frac{x^2}{2x+1}$, for $x \neq -\frac{1}{2}$.

(a) Show that $\frac{dy}{dx} = \frac{2x^2+2x}{(2x+1)^2}$.

(b) Find the coordinates of the stationary points of C. [**ans:** (0,0) and (-1,-1)]

Q2. The curve C has the equation $y = \frac{-x}{1+\ln x}$. Show that the curve has a single stationary point and find its coordinates.

[**ans:** (1,-1)]

Q3. The curve C has equation $y = (x-1)(x-2) + \ln x$, for $x > 0$.

(a) Show that one turning point of C is $(\frac{1}{2}, \frac{3}{4} - \ln 2)$ and find the other. [**ans:** (1,0)]

(b) Determine the nature of the turning point at $x = \frac{1}{2}$. [**ans:** Local maximum]

Q4. The curve C has equation $y = \ln x - \frac{x}{4}$, for $x > 0$. Find the exact coordinates of the turning point of C and determine its nature.

[**ans:** $(4, \ln 4 - 1)$, which is a maximum]

Q5. The curve C is given by $y = \frac{kx^2 - a}{kx^2 + a}$, where k and a are non-zero constants.

(a) Find a simplified expression for $\frac{dy}{dx}$. $\left[\textbf{ans: } \frac{dy}{dx} = \frac{4akx}{(kx^2+a)^2}\right]$
(b) Hence show that C has a single turning point and state its coordinates. [**ans:** (0, -1)]

Q6. A curve C is given by $y = x^2e^x$.

(a) Find the exact coordinates of the stationary points of C. [**ans:** (0,0) and $(-2, 4e^{-2})$]
(b) By considering the sign of $\frac{d^2y}{dx^2}$, determine their nature. [**ans:** (0,0) min, $(-2, 4e^{-2})$ max]

Q7. The curve C has equation $f(x) = (2x-1)e^{-2x}$.

(a) Find an expression for $f'(x)$. [**ans:** $f'(x) = (4-4x)e^{-2x}$]
(b) Show that $f''(x) = 4(2x-3)e^{-2x}$.
(c) Find the exact coordinates of the stationary point of C and determine its nature.
[**ans:** $(1, e^{-2})$, which is a maximum]

Q8. For the curve $y = 12\ln x - x^{\frac{3}{2}}$, where $x > 0$, determine the range of values of x for which y is decreasing.
[**ans:** $x > 4$]

Q9. A curve C has equation $y = e^{2x}(x^2 - 4x - 2)$.
(a) Show that $\frac{dy}{dx} = 2e^{2x}(x^2 - 3x - 4)$.
(b) Show further that $\frac{d^2y}{dx^2} = 2e^{2x}(2x^2 - 4x - 11)$.
(c) Find the exact coordinates of the stationary points of C and determine their nature.
[**ans:** Local min at $(4, -2e^8)$, local max at $(-1, 3e^{-2})$]

Q11. The curve C has equation $f(x) = ax^3 + 15x^2 - 39x + b$. The point $(2, 10)$ lies on C and the gradient at this point is -3.

(a) (i) Show that $a = -2$. (ii) Find the value of b. [**ans:** $b = 44$]
(b) Hence show that C has no stationary points.
(c) Write $f(x)$ in the form $(x-4)Q(x)$. [**ans:** $Q(x) = -2x^2 + 7x - 11$]
(d) Deduce the coordinates where $y = f(0.2x)$ intersects the axes. [**ans:** (0,44) and (20,0)]

Q12. A cubic curve $y = g(x)$ passes through the origin and has a stationary point at $(2, 9)$. The coefficient of its x^3 term is equal to the coefficient of its x term.

(a) Find $g(x)$. [**ans:** $g(x) = -3x^3 + \frac{39}{4}x^2 - 3x$]
(b) Prove that the stationary point at $(2, 9)$ is a maximum.

Q13. The curve C has equation $y = 12\sqrt{x} - x^2 - 10$, for $x > 0$.

(a) Find the coordinates of the turning point on C. [**ans:** $(\sqrt[3]{9}, 9\sqrt[3]{3} - 10)$]
(b) Find $\frac{d^2y}{dx^2}$. $\left[\textbf{ans: } \frac{d^2y}{dx^2} = -3x^{-3/2} - 2\right]$
(c) State the nature of the turning point. [**ans:** Maximum]

Q14. The equation of a curve C is $y = \frac{x}{x^2+9}$.

(a) Find the coordinates of the stationary points of C. [**ans:** $(3, 1/6)$ and $(-3, -1/6)$]
(b) Evaluate $\frac{d^2y}{dx^2}$ at each stationary point. [**ans:** At $x = 3$, $-1/54$; at $x = -3$, $1/54$]

2.6 Partial derivatives of multi-variable functions

So far we have dealt with functions($f(x)$) which have one independent input variable which can be interpreted graphically as a planar curve. Now we will be dealing with functions($f(x,y,z)$) with multiple independent input variables, these functions can be interpreted graphically as a surface.

In order to find the rate of change of multi-variable functions, a process called partial differentiation is used. This is done by holding all but one of the variables constant and finding the rate of change of the function with respect to the one remaining variable.

A function with two independent variables f(x,y) has two partial derivatives, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Likewise a function of three variables f(x,y,z) has three partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$, and so on for functions with more than three variables.

When differentiating a function with several variables $f(x,y,z,a,b,...)$ the partial derivative of f with respect to a (for example) is denoted by $\frac{\partial f}{\partial a}$ and is obtained by differentiating $f(x,y,z,a,b,...)$ with respect to a in the usual way but treating all the other variables as if they were constants.

Example 2.11 Finding First-Order Partial Derivatives

Problem: Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the function:

$$f(x,y) = x^2y + 3xy^2 + y^3$$

Finding $\frac{\partial f}{\partial x}$: (Treat y as a constant)

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2y + 3xy^2 + y^3)$$

$$\frac{\partial f}{\partial x} = (2x)y + (3)y^2 + 0$$

$$\frac{\partial f}{\partial x} = 2xy + 3y^2$$

Finding $\frac{\partial f}{\partial y}$: (Treat x as a constant)

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2y + 3xy^2 + y^3)$$

$$\frac{\partial f}{\partial y} = x^2(1) + 3x(2y) + 3y^2$$

$$\frac{\partial f}{\partial y} = x^2 + 6xy + 3y^2$$

The partial derivatives are:
$\frac{\partial f}{\partial x} = 2xy + 3y^2$
$\frac{\partial f}{\partial y} = x^2 + 6xy + 3y^2$

■

Q1. For the function $f(x,y,z) = 4x^3y^2 - e^{y^4} + \frac{z^2}{x^2} + 4y - x^{16}$, find:

(a) $\dfrac{\partial f}{\partial x}$ $\left[\textbf{ans: } 12x^2y^2 - \dfrac{2z^2}{x^3} - 16x^{15}\right]$

(b) $\dfrac{\partial f}{\partial y}$ $\left[\textbf{ans: } 8x^3y - 4y^3e^{y^4} + 4\right]$

(c) $\dfrac{\partial f}{\partial z}$ $\left[\textbf{ans: } \dfrac{2z}{x^2}\right]$

Q2. For the function $f(x,y,z) = \cos(x^2+2y) - e^{4x-z^4y+y^3}$, find:

(a) $\dfrac{\partial f}{\partial x}$ $\left[\textbf{ans: } -2x\sin(x^2+2y) - 4e^{4x-z^4y+y^3}\right]$

(b) $\dfrac{\partial f}{\partial y}$ $\left[\textbf{ans: } -2\sin(x^2+2y) + (z^4-3y^2)e^{4x-z^4y+y^3}\right]$

(c) $\dfrac{\partial f}{\partial z}$ $\left[\textbf{ans: } 4yz^3e^{4x-z^4y+y^3}\right]$

Q3. For the function $f(p,t,u,v) = 8u^2t^3p - \sqrt{vp^2t^{-5}} + 2u^2t + 3p^4 - v$, find:

(a) $\dfrac{\partial f}{\partial p}$ $\left[\textbf{ans: } 8u^2t^3 - v^{1/2}t^{-5/2} + 12p^3\right]$

(b) $\dfrac{\partial f}{\partial t}$ $\left[\textbf{ans: } 24u^2t^2p + \frac{5}{2}v^{1/2}pt^{-7/2} + 2u^2\right]$

(c) $\dfrac{\partial f}{\partial u}$ $\left[\textbf{ans: } 16ut^3p + 4ut\right]$

(d) $\dfrac{\partial f}{\partial v}$ $\left[\textbf{ans: } -\frac{1}{2}v^{-1/2}pt^{-5/2} - 1\right]$

Q4. For $f(u,v) = u^2\sin(u+v^3) - \sec(4u)\arctan(2v)$, find:

(a) $\dfrac{\partial f}{\partial u}$ $\left[\textbf{ans: } 2u\sin(u+v^3) + u^2\cos(u+v^3) - 4\sec(4u)\tan(4u)\arctan(2v)\right]$

(b) $\dfrac{\partial f}{\partial v}$ $\left[\textbf{ans: } 3v^2u^2\cos(u+v^3) - \dfrac{2\sec(4u)}{1+4v^2}\right]$

Q5. For $f(x,z) = e^{-x}\dfrac{\sqrt{4z^4+x^2}}{x} + \dfrac{2x+3z}{3x}$, find:

(a) $\dfrac{\partial f}{\partial x}$ $\left[\textbf{ans: } -e^{-x}\dfrac{\sqrt{4z^4+x^2}}{x} - \dfrac{4z^4e^{-x}}{x^2\sqrt{4z^4+x^2}} - \dfrac{z}{x^2}\right]$

(b) $\dfrac{\partial f}{\partial z}$ $\left[\textbf{ans: } \dfrac{8z^3e^{-x}}{x\sqrt{4z^4+x^2}} + \dfrac{1}{x}\right]$

Q6. For $g(s,t,v) = t^2\ln(s+2t) - \ln(3v)(s^3+t^2-4v)$, find:

(a) $\dfrac{\partial g}{\partial s}$ $\left[\textbf{ans: } \dfrac{t^2}{s+2t} - 3s^2\ln(3v)\right]$

(b) $\dfrac{\partial g}{\partial t}$ $\left[\textbf{ans: } 2t\ln(s+2t) + \dfrac{2t^2}{s+2t} - 2t\ln(3v)\right]$

(c) $\dfrac{\partial g}{\partial v}$ $\left[\textbf{ans: } -\dfrac{s^3+t^2-4v}{v} + 4\ln(3v)\right]$

Q7. For the function $R(x,y) = \dfrac{x^2}{y^2+1} - \dfrac{y^2}{x^2+y}$, find:

(a) An expression for $\dfrac{\partial R}{\partial x}$ $\left[\textbf{ans: } \dfrac{2x}{y^2+1} + \dfrac{2xy^2}{(x^2+y)^2}\right]$

(b) An expression for $\dfrac{\partial R}{\partial y}$ $\left[\textbf{ans: } -\dfrac{2x^2y}{(y^2+1)^2} - \dfrac{y(2x^2+y)}{(x^2+y)^2}\right]$

Q8. For the function $f(p,t,r) = \dfrac{p^2(r+1)}{t^3} + pre^{2p+3r+4t}$, find:

(a) An expression for $\dfrac{\partial f}{\partial p}$ $\left[\textbf{ans: } \dfrac{2p(r+1)}{t^3} + (r+2pr)e^{2p+3r+4t}\right]$

(b) An expression for $\dfrac{\partial f}{\partial t}$ $\left[\textbf{ans: } -\dfrac{3p^2(r+1)}{t^4} + 4pre^{2p+3r+4t}\right]$

(c) An expression for $\dfrac{\partial f}{\partial r}$ $\left[\textbf{ans: } \dfrac{p^2}{t^3} + (p+3pr)e^{2p+3r+4t}\right]$

Q9. For the equation $x^2\sin(y^3) + xe^{3z} - \cos(z^2) = 3y - 6z + 8$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

$$\left[\textbf{ans: } \frac{\partial z}{\partial x} = -\frac{2x\sin(y^3)+e^{3z}}{3xe^{3z}+2z\sin(z^2)+6}, \quad \frac{\partial z}{\partial y} = \frac{3(1-x^2y^2\cos(y^3))}{3xe^{3z}+2z\sin(z^2)+6}\right]$$

Q10. The function z depends on u and v as shown below. Find simplified expressions for $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$, in terms of x and y.

$$z = (2x+3y)^2, \quad u = x^2+y^2 \quad \text{and} \quad v = x+2y.$$

$$\left[\textbf{ans: } \frac{\partial z}{\partial u} = \frac{2x+3y}{2x-y}, \quad \frac{\partial z}{\partial v} = \frac{2(2x+3y)(3x-2y)}{2x-y}\right]$$

Q11. Given $f(x,y,z) = 2x + y^2 + xz$, where $x = 2t$, $y = t^2$, and $z = 3$.

(a) Use the chain rule for partial derivatives to find $\dfrac{df}{dt}$. [**ans:** $10+4t^3$]

(b) Verify your answer by first substituting for x,y,z and then differentiating.

Q12. The function φ is shown below, where $x = 3t$, $y = t^2$, and $z = \frac{1}{t}$.

$$\varphi(x,y,z) = x^2 + y^2 + tz + t, \quad t \neq 0,$$

(a) Find an expression for $\dfrac{d\varphi}{dt}$, in terms of t. [**ans:** $4t^3+18t+1$]

(b) Verify the answer obtained in part (a) by a method not involving partial differentiation.
[**ans:** Direct substitution gives $\varphi = t^4 + 9t^2 + t + 1$, whose derivative is the same.]

Q13. Given the transformations $x = r\cos\theta$ and $y = r\sin\theta$, find expressions for the following partial derivatives in terms of r and θ.

(a) $\dfrac{\partial r}{\partial x}$ [**ans:** $\cos\theta$]

(b) $\dfrac{\partial r}{\partial y}$ [**ans:** $\sin\theta$]

(c) $\dfrac{\partial \theta}{\partial x}$ [**ans:** $-\dfrac{\sin\theta}{r}$]

(d) $\dfrac{\partial \theta}{\partial y}$ [**ans:** $\dfrac{\cos\theta}{r}$]

2.7 Chapter Review

Review Exercise 3 Mixed Questions

Problem 2.7.1 The point P, where $x=\pi$, lies on the curve with equation $f(x)=e^x\sin 2x$. $0\le x<2\pi$. Show that an equation of the normal to the curve at P is given by $x+2ye^{\pi}=\pi$.

Problem 2.7.2 Differentiate each of the following expressions with respect to x, simplifying your answers where possible.

(a) $y=\dfrac{1}{\sqrt{1-2x}}$

(b) $y=e^{3x}(\sin x+\cos x)$

(c) $y=\dfrac{\ln x}{x^2}$

Problem 2.7.3 A curve has the equation $y=x\ln(1+x^2)$. Show that an equation of the tangent to the curve at the point where $x=1$ is given by $y=x(1+\ln 2)-1$.

Problem 2.7.4 The equation of a curve C is given by $y=e^{2x}(\cos x+\sin x)$.

(a) Find an expression for $\dfrac{dy}{dx}$.

(b) Show that this can be simplified to $\dfrac{dy}{dx}=e^{2x}(\sin x+3\cos x)$.

(c) Hence, show that the x-coordinates of the turning points of C satisfy the equation $\tan x=-3$.

Problem 2.7.5 Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ for the following functions:

(a) $f(x,y)=(x^2-1)(y-2)$

(b) $f(x,y)=e^{x-y-1}$

(c) $f(x,y)=e^{-x}\sin(x-y)$

Problem 2.7.6 Given that $y=\dfrac{3\sin\theta}{2(\sin\theta+\cos\theta)}$ for $-\frac{\pi}{4}<\theta<\frac{3\pi}{4}$.

show that $\dfrac{dy}{d\theta}=\dfrac{A}{1+\sin 2\theta}$, where A is a rational constant to be found.

■

Review Exercise 4 Exam-Style Questions

Problem 2.7.7 Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the following functions:

(a) $f(x,y) = (xy+1)^2$

(b) $f(x,y) = \frac{1}{x-y}$

(c) $f(x,y) = \ln(x-y)$

(d) $f(x,y) = \sin^2(x+3y)$

(e) $f(x,y) = x^{-y}$

Problem 2.7.8 The curve C has equation

$$y = \frac{x^3\left(5\sqrt{x}-128\right)}{\sqrt{x}}, \quad x > 0.$$

(a) Determine expressions for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, and $\frac{d^3y}{dx^3}$.

(b) Show that the y-coordinate of the stationary point of C is $-k\sqrt[3]{4}$, where k is a positive integer.

(c) Evaluate $\frac{d^2y}{dx^2}$ at the stationary point of C. Give the answer in terms of $\sqrt[3]{2}$.

(d) Find the value of $\frac{d^3y}{dx^3}$ at the point on C where $\frac{d^2y}{dx^2} = 0$.

Problem 2.7.9 A curve is defined by the parametric equations

$$x = t^3 - 3t, \quad y = t^2, \quad t \in \mathbb{R}.$$

The point P on the curve is where $t = 2$.

(a) Find the equation of the normal to the curve at point P.

(b) Find an expression for $\frac{d^2y}{dx^2}$ in terms of t.

(c) Hence, determine the concavity of the curve at point P.

Problem 2.7.10 A curve is defined implicitly by the equation $x^2 - xy + y^2 = 3$.

(a) Find an expression for $\frac{dy}{dx}$.

(b) Find the coordinates of all points on the curve where the tangent is horizontal.

(c) Find the coordinates of all points on the curve where the tangent is vertical.

■

Summary 2 Differentiation

Basic Rules:

- **Derivative of a constant:** $\frac{d}{dx}(c) = 0$
- **Power Rule:** $\frac{d}{dx}(x^n) = nx^{n-1}$
- **Sum/Difference:** $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
- **Constant Multiple:** $\frac{d}{dx}[cf(x)] = cf'(x)$

Rules for Combining Functions:

- **Product Rule:** $\frac{d}{dx}[f(x)g(x)] = f'g + fg'$
- **Quotient Rule:** $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$
- **Chain Rule:** $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Derivatives of Standard Functions:

- **Exponential:** $\frac{d}{dx}(e^{kx}) = ke^{kx}$, $\frac{d}{dx}(a^x) = a^x \ln a$
- **Logarithmic:** $\frac{d}{dx}(\ln x) = \frac{1}{x}$, $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$
- **Trigonometric:** $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$, $\frac{d}{dx}(\tan x) = \sec^2 x$

Advanced Techniques:

- **Parametric:** If $x = x(t)$ and $y = y(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.
- **Implicit:** For an equation relating x and y, differentiate both sides with respect to x, treating y as a function of x, and solve for $\frac{dy}{dx}$.

Partial Differentiation: The derivative of a multivariable function with respect to one variable, while all other variables are treated as constants.

Chapter Checklist

- ☐ I can apply the power, product, quotient, and chain rules correctly.
- ☐ I can differentiate standard exponential, logarithmic, and trigonometric functions.
- ☐ I can find the equations of tangents and normals to a curve.
- ☐ I can find and classify the stationary points of a function.
- ☐ I can use implicit and parametric differentiation to find derivatives.
- ☐ I can find the first-order partial derivatives of a multivariable function.

3. Series

Series can be used to represent complicated functions in alegbraic polynomial form. This is only possible if the function can be differentiated as many times as possible and it must be smooth; which means that it is valid at for our point of interest.

3.1 Maclaurin Series

If $f(x)$ can be differentiated as often as required:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots = \sum_{p=0}^{\infty} \frac{x^p}{p!} f^{(p)}(0)$$

This is the Maclaurin expansion of $f(x)$. The Maclaurin series is widely used in engineering for signal processing, analysing small angle approximations, simplifying nonlinear equations, and solving differential equations.

Example 3.1 **Finding a Maclaurin Series from First Principles**

Problem: Find the Maclaurin series for $f(x) = \ln(1+x)$ up to and including the term in x^4.

Solution:

Step 1: Find the first four derivatives of $f(x)$ and evaluate each at $x = 0$.

$$f(x) = \ln(1+x) \implies f(0) = \ln(1) = 0$$

$$f'(x) = \frac{1}{1+x} \implies f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \implies f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \implies f'''(0) = 2$$

$$f^{(4)}(x) = -\frac{6}{(1+x)^4} \implies f^{(4)}(0) = -6$$

Step 2: Substitute these values into the Maclaurin series formula.

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

Step 3: Insert the calculated values and simplify each term.

$$f(x) = 0 + (1)x + \frac{-1}{2}x^2 + \frac{2}{6}x^3 + \frac{-6}{24}x^4 + \dots$$

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots}$$

Q1. Find the Maclaurin expansion of $f(x) = (1-x)^2 \ln(1-x)$ up to and including the term in x^3.

$$\left[\textbf{ans: } f(x) = -x + \frac{3}{2}x^2 - \frac{1}{3}x^3 + \dots\right]$$

Q2. Find the Maclaurin expansion of $f(x) = e^{-2x}\cos 4x$ up to and including the term in x^4.

$$\left[\textbf{ans: } 1 - 2x - 6x^2 + \frac{44}{3}x^3 - \frac{14}{3}x^4 + \dots\right]$$

Q3. Use standard results to find the series expansion of $y = e^{2x}\sin 3x$ up to and including the term in x^4.

$$\left[\textbf{ans: } 3x + 6x^2 + \frac{3}{2}x^3 - 5x^4 + \dots\right]$$

Q4. Given $f(x) = \sin[\ln(1+x)]$, for $x > -1$:

(a) Show that $(1+x)^2 f''(x) + (1+x)f'(x) + f(x) = 0$.

(b) Hence find the first three non-zero terms of the Maclaurin expansion of $f(x)$.

$$\left[\textbf{ans: } x-\frac{1}{2}x^2+\frac{1}{6}x^3\right]$$

Q5. Given $y=\ln(4+3x)$, for $x>-\frac{4}{3}$:

(a) Find simplified expressions for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, and $\frac{d^3y}{dx^3}$.

$$\left[\textbf{ans: } \frac{3}{4+3x},\quad -\frac{9}{(4+3x)^2},\quad \frac{54}{(4+3x)^3}\right]$$

(b) Hence, find the first four terms in the Maclaurin expansion of y.

$$\left[\textbf{ans: } \ln 4+\frac{3}{4}x-\frac{9}{32}x^2+\frac{9}{64}x^3+\dots\right]$$

Q6. The first non-zero term in the Maclaurin expansion of $e^{mx}-(1+4x)^n$ is $-4x^2$, where m and n are non-zero constants. Find the coefficient of x^3 in this expansion.

$$\left[\textbf{ans: } \frac{56}{3}\right]$$

3.2 Taylor Series

If the function $f(x)$ can be differentiated as often as required at $x = x_0$ then:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \cdots$$

This is called the **Taylor series** of $f(x)$ about the point x_0.

The Maclaurin series is equivalent to the Taylor series of that function about the point $x = 0$. The Taylor series is used in engineering for modeling stresses in structures, analysing circuit responses, and approximating fluid and thermodynamic properties.

Example 3.2 Finding a Taylor Series

Problem: Find the Taylor series for $f(x) = \sqrt{x}$ about the point $a = 4$, up to $(x-4)^2$.

Solution:

Step 1: Find the first two derivatives of $f(x)$ and evaluate each at $a = 4$.

$$f(x) = x^{1/2} \implies f(4) = \sqrt{4} = 2$$

$$f'(x) = \frac{1}{2}x^{-1/2} \implies f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} \implies f''(4) = -\frac{1}{4(4)^{3/2}} = -\frac{1}{32}$$

Step 2: State the Taylor series formula up to the second-order term.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$$

Step 3: Substitute the calculated values with $a = 4$ and simplify.

$$f(x) = 2 + \left(\frac{1}{4}\right)(x-4) + \frac{-1/32}{2}(x-4)^2 + \ldots$$

$$f(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \ldots$$

$$\boxed{\sqrt{x} \approx 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2}$$

■

Q1. Find the first four terms in the Taylor expansion of $y = \frac{1}{\sqrt{x}}$ about $x = 1$.

$$\left[\textbf{ans: } 1 - \frac{1}{2}(x-1) + \frac{3}{8}(x-1)^2 - \frac{5}{16}(x-1)^3 + \ldots\right]$$

Q2. Let $y = \tan x$.

(a) Show that $\frac{d^3y}{dx^3} = 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2$.

(b) Determine the first four terms in the Taylor expansion of $\tan x$ in ascending powers of $\left(x-\frac{\pi}{4}\right)$.

$$\left[\textbf{ans: } 1+2\left(x-\tfrac{\pi}{4}\right)+2\left(x-\tfrac{\pi}{4}\right)^2+\frac{8}{3}\left(x-\tfrac{\pi}{4}\right)^3+\dots\right]$$

Q3. Let $y=\tan^2 x$.

(a) Show that $\dfrac{d^4y}{dx^4}=120\sec^6 x-120\sec^4 x+16\sec^2 x$.

(b) Determine the first five terms in the Taylor expansion of $\tan^2 x$ in ascending powers of $\left(x-\frac{\pi}{3}\right)$.

$$\left[\textbf{ans: } 3+8\sqrt{3}h+40h^2+\frac{176\sqrt{3}}{3}h^3+\frac{728}{3}h^4+\dots \text{ where } h=x-\tfrac{\pi}{3}\right]$$

Q4. Let $f(x)=\cos x$.

(a) Find the first four terms in the Taylor expansion of $f(x)$ in ascending powers of $\left(x-\frac{\pi}{6}\right)$.

$$\left[\textbf{ans: } \frac{\sqrt{3}}{2}-\frac{1}{2}\left(x-\tfrac{\pi}{6}\right)-\frac{\sqrt{3}}{4}\left(x-\tfrac{\pi}{6}\right)^2+\frac{1}{12}\left(x-\tfrac{\pi}{6}\right)^3+\dots\right]$$

(b) Use this expansion to find an approximation for $\cos\left(\frac{\pi}{4}\right)$.

$$\left[\textbf{ans: } \cos\left(\tfrac{\pi}{4}\right)\approx\frac{\sqrt{3}}{2}-\frac{\pi}{24}-\frac{\sqrt{3}\pi^2}{576}+\frac{\pi^3}{20736}\right]$$

Q5. Find the first three non-zero terms in the Taylor expansion of $f(x)=\sin 2x$ centred at $x=\frac{\pi}{4}$.

$$\left[\textbf{ans: } 1-2\left(x-\tfrac{\pi}{4}\right)^2+\frac{2}{3}\left(x-\tfrac{\pi}{4}\right)^4+\dots\right]$$

Q6. Given the differential equation $\dfrac{d^2y}{dx^2}=e^x\left(2y\dfrac{dy}{dx}+y^2+1\right)$ with initial conditions $y(0)=1$ and $\left.\frac{dy}{dx}\right|_{x=0}=2$.

(a) Show that $\dfrac{d^3y}{dx^3}=e^x\left[2y\dfrac{d^2y}{dx^2}+2\left(\dfrac{dy}{dx}\right)^2+ky\dfrac{dy}{dx}+y^2+1\right]$ and find the value of k.

[**ans:** $k=4$]

(b) Find the series solution for y in ascending powers of x, up to and including the term in x^3.

[**ans:** $y=1+2x+3x^2+5x^3+\dots$]

3.3 Chapter Review

Review Exercise 5 Mixed Questions

Problem 3.3.1 By multiplying the standard series for e^{x^2} and $\cos(3x)$, find the Maclaurin expansion of $f(x) = e^{x^2}\cos(3x)$ up to and including the term in x^4.

Problem 3.3.2

(a) Find the first three terms of the Taylor series for $f(x) = \sqrt[3]{x}$ in ascending powers of $(x-8)$.

(b) Use your expansion to find an approximation for $\sqrt[3]{8.1}$, giving your answer to 5 decimal places.

Problem 3.3.3 The Maclaurin series for the function $f(x) = \ln(1+ax) - \sin(bx)$ is given by $f(x) = 2x - \frac{11}{2}x^2 + \dots$. Find the exact values of the constants a and b.

Problem 3.3.4 A function $y = f(x)$ satisfies the differential equation $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$, with initial conditions $y(0) = 1$ and $y'(0) = -1$. By differentiating the equation repeatedly, find the Maclaurin series for y up to and including the term in x^4.

■

Review Exercise 6 Challenging Questions

Problem 3.3.5 A function $y = f(x)$ satisfies the differential equation shown below, with the initial condition $y(0) = 1$.

$$\frac{dy}{dx} = 1 + x + y^2$$

(a) By differentiating the equation, find the values of $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ at $x = 0$.

(b) Hence, find the Maclaurin series for y up to and including the term in x^3.

Problem 3.3.6

(a) Find the first three non-zero terms of the Maclaurin series for $f(x) = \frac{\sin(x)}{1-x}$.

(b) Use your series from part (a) to find an approximation for the value of the integral, giving your answer to 4 decimal places.

$$\int_0^{0.5} \frac{\sin(x)}{1-x}\,dx$$

Problem 3.3.7 It is given that, for small values of x,

$$e^{ax}\cos(bx) \approx 1 + 2x - 2x^2$$

where a and b are non-zero constants.

(a) By using standard Maclaurin expansions, find the values of a and b.

(b) Hence, find the term in x^3 for the expansion.

■

Summary 3 Series

Maclaurin Series A power series expansion of a function about the point $a = 0$. It is derived by repeatedly differentiating the function and evaluating at zero.

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Taylor Series The generalisation of the Maclaurin series, providing an expansion of a function $f(x)$ about any point $x = a$.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Standard Expansions

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}, \quad |x| < 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

Chapter Checklist

- ☐ I can find a Maclaurin series for a function by repeated differentiation.
- ☐ I can find a Taylor series for a function about a point $x = a$.
- ☐ I can use standard series expansions to find new series by substitution, addition, and multiplication.
- ☐ I can use a series expansion to find an approximate value for a function.
- ☐ I can find the first few terms of a series solution to a differential equation.

■

4. Integration

Integration is a mathematical technique used to determine the accumulation of quantities and the area under curves. Essentially, it measures the total value accumulated by a function over an interval.In engineering applications, integration is used to calculate quantities such as displacement from velocity, energy consumption over time, total heat transfer in thermal systems, and the work done by a variable force. It can also model the flow of fluids over surfaces, predict the total stress distribution in materials, or determine electrical charge over time in circuits.

Function	**Integral**
Basic & Exponential	
k	$kx+c$
x^n	$\frac{x^{n+1}}{n+1}+c \quad (n \neq -1)$
$\frac{1}{x}$	$\ln\lvert x\rvert+c$
e^{kx}	$\frac{1}{k}e^{kx}+c$
Hyperbolic Functions	
$\sinh x$	$\cosh x+c$
$\cosh x$	$\sinh x+c$
$\tanh x$	$\ln(\cosh x)+c$
$\coth x$	$\ln\lvert\sinh x\rvert+c$

Function	**Integral**
Trigonometric Functions	
$\sin x$	$-\cos x+c$
$\cos x$	$\sin x+c$
$\tan x$	$\ln\lvert\sec x\rvert+c$
$\sec x$	$\ln\lvert\sec x+\tan x\rvert+c$
$\csc x$	$-\ln\lvert\csc x+\cot x\rvert+c$
$\cot x$	$\ln\lvert\sin x\rvert+c$
Inverse Trig	
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)+c$
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right)+c$
$\frac{1}{\sqrt{x^2\pm a^2}}$	$\ln\left\lvert x+\sqrt{x^2\pm a^2}\right\rvert+c$

4.1 Integration of Standard Functions

4.1.1 Polynomials Functions

Integrating polynomials is the reverse process of differentiation and relies on applying the power rule to each term. The ability to integrate term-by-term is due to two fundamental properties of integrals:

The Constant Multiple Rule: $\int kf(x)\,dx = k\int f(x)\,dx$

The Sum/Difference Rule: $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

These rules allow us to break down a complex polynomial into simpler parts. The core formula used for each part is the Power Rule for Integration. The indefinite integral of x^n is given by the formula:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad (\text{for } n \neq -1)$$

In simple terms: you add one to the exponent and then divide by the new exponent. The constant of integration, C, must always be included because the derivative of any constant is zero.

Example 4.1 Integration of a Polynomial

Problem: Find the indefinite integral of the function:

$$\int \left(3x^4 - 5x^3 + 2x - 7\right)\,dx$$

Solution:

Step 1: Recall the power rule for integration: $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$

Step 2: Apply the power rule to each term of the polynomial.

$$\int \left(3x^4 - 5x^3 + 2x - 7\right)\,dx$$
$$= 3\left(\frac{x^5}{5}\right) - 5\left(\frac{x^4}{4}\right) + 2\left(\frac{x^2}{2}\right) - 7(x) + C$$

Step 3: Simplify the expression.

$$= \frac{3}{5}x^5 - \frac{5}{4}x^4 + x^2 - 7x + C$$

$$\boxed{\int \left(3x^4 - 5x^3 + 2x - 7\right)\,dx = \frac{3}{5}x^5 - \frac{5}{4}x^4 + x^2 - 7x + C}$$

■

Q1. Find the indefinite integral of $f(x) = 2x^2$. $\left[\textbf{ans: } \frac{2}{3}x^3 + C\right]$

Q2. Find the indefinite integral of $f(x) = x^3 + 7$. $\left[\textbf{ans: } \frac{x^4}{4} + 7x + C\right]$

Q3. Find the indefinite integral of $f(x) = 5x^4 + 2x^3 + 4x + 7$. $\left[\textbf{ans: } x^5 + \frac{1}{2}x^4 + 2x^2 + 7x + C\right]$

Q4. Find the indefinite integral of $f(x) = \frac{3}{2x^2} + \frac{7}{3x^3} + 18x + 27$.
$\left[\textbf{ans: } -\frac{3}{2x} - \frac{7}{6x^2} + 9x^2 + 27x + C\right]$

Q5. Find the indefinite integral of $f(t) = 113$. $[\textbf{ans: } 113t + C]$

Q6. Find the indefinite integral of $f(r) = \pi r^2$. $\left[\textbf{ans: } \frac{\pi r^3}{3} + C\right]$

4.1.2 Trigonometric Functions

The integration of trigonometric functions relies on recognising a set of standard results. As with polynomials, more complex expressions involving sums and constant multiples of these functions can be integrated term-by-term.

The following are the standard indefinite integrals for common trigonometric functions. In these formulae, k is a non-zero constant.

- $\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + C$
- $\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + C$
- $\int \tan(kx)\,dx = \frac{1}{k}\ln|\sec(kx)| + C$
- $\int \cot(kx)\,dx = \frac{1}{k}\ln|\sin(kx)| + C$
- $\int \sec(kx)\,dx = \frac{1}{k}\ln|\sec(kx) + \tan(kx)| + C$
- $\int \csc(kx)\,dx = -\frac{1}{k}\ln|\csc(kx) + \cot(kx)| + C$
- $\int \tan^2(kx)\,dx = \frac{1}{k}\tan(kx) - x + C$
- $\int \cot^2(kx)\,dx = -\frac{1}{k}\cot(kx) - x + C$
- $\int \sec^2(kx)\,dx = \frac{1}{k}\tan(kx) + C$
- $\int \csc^2(kx)\,dx = -\frac{1}{k}\cot(kx) + C$

Example 4.2 Integration of Trigonometric Functions

Problem: Find the indefinite integral of the function:

$$\int \left(\sin x + 2\cos x - 3\sec^2 x\right)\,dx$$

Solution:

Step 1: Recall the standard integrals for basic trigonometric functions.

$$\int \sin x\, dx = -\cos x, \quad \int \cos x\, dx = \sin x, \quad \int \sec^2 x\, dx = \tan x$$

Step 2: Apply these rules to integrate the function term-by-term.

$$\int \left(\sin x + 2\cos x - 3\sec^2 x\right)\, dx$$

$$= \int \sin x\, dx + 2\int \cos x\, dx - 3\int \sec^2 x\, dx$$

Step 3: Substitute the results of the integrals and add the constant of integration.

$$= (-\cos x) + 2(\sin x) - 3(\tan x) + C$$

$$= -\cos x + 2\sin x - 3\tan x + C$$

$$\boxed{\int \left(\sin x + 2\cos x - 3\sec^2 x\right)\, dx = -\cos x + 2\sin x - 3\tan x + C}$$

■

Q1. Carry out the following integrations:

(a) $\int 3\sin^2 x\, dx$ $\left[\textbf{ans: } \frac{3}{2}x - \frac{3}{4}\sin 2x + C\right]$

(b) $\int 4\cos^2 x\, dx$ $[\textbf{ans: } 2x + \sin 2x + C]$

(c) $\int 3\sin x\cos x\, dx$ $\left[\textbf{ans: } -\frac{3}{4}\cos 2x + C\right]$

(d) $\int (2 - 3\sin x)^2\, dx$ $\left[\textbf{ans: } \frac{17}{2}x + 12\cos x - \frac{9}{4}\sin 2x + C\right]$

(e) $\int (1 - \cos 2x)^2\, dx$ $\left[\textbf{ans: } \frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x + C\right]$

Q2. Find the following integrals:

(a) $\int 2\tan^2 x\, dx$ $[\textbf{ans: } 2\tan x - 2x + C]$

(b) $\int 5\cot^2 x\, dx$ $[\textbf{ans: } -5\cot x - 5x + C]$

(c) $\int (2\tan x - \cot x)^2\, dx$ $[\textbf{ans: } 4\tan x - \cot x - 9x + C]$

(d) $\int \frac{4\sin x}{\cos^2 x}\, dx$ $[\textbf{ans: } 4\sec x + C]$

(e) $\int \frac{\cos x}{3\sin^2 x}\, dx$ $\left[\textbf{ans: } -\frac{1}{3}\csc x + C\right]$

Q3. Carry out the following integrations:

(a) $\int (2+2\tan^2 x)\,dx$ [**ans:** $2\tan x + C$]

(b) $\int \frac{1+\cos x}{\sin^2 x}\,dx$ [**ans:** $-\cot x - \csc x + C$]

(c) $\int \frac{(1+\cos x)^2}{\sin^2 x}\,dx$ [**ans:** $-2\cot x - 2\csc x - x + C$]

(d) $\int 3\cot^2 x\,dx$ [**ans:** $-3\cot x - 3x + C$]

Q4. Solve the following integrals:

(a) $\int \frac{\cos 2x}{1-\cos^2 2x}\,dx$ $\left[\textbf{ans: } -\frac{1}{2}\csc 2x + C\right]$

(b) $\int \cot 2x\,dx$ $\left[\textbf{ans: } \frac{1}{2}\ln|\sin 2x| + C\right]$

(c) $\int \sin 2x \sec x\,dx$ [**ans:** $-2\cos x + C$]

(d) $\int \frac{1}{\sin x \cos^2 x}\,dx$ [**ans:** $\sec x + \ln|\tan(x/2)| + C$]

(e) $\int \frac{1}{\sec x - 1}\,dx$ [**ans:** $-x - \cot x - \csc x + C$]

(f) $\int (1-\cot^2 x)\,dx$ [**ans:** $2x + \cot x + C$]

Q5. Evaluate the following definite integrals:

(a) $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$ $\left[\textbf{ans: } \frac{\pi}{4}\right]$

(b) $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$ $\left[\textbf{ans: } \frac{\pi}{4}\right]$

(c) $\int_0^{\frac{\pi}{2}} (2\sin x - 3\cos x)^2\,dx$ $\left[\textbf{ans: } \frac{13\pi}{4} - 6\right]$

(d) $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1-2\cos x)^2\,dx$ [**ans:** $4\pi + 3\sqrt{3}$]

(e) $\int_0^{\frac{\pi}{4}} \tan^2 x\,dx$ $\left[\textbf{ans: } 1 - \frac{\pi}{4}\right]$

4.1.3 Exponential Functions

The integration of exponential functions is a straightforward process that directly reverses their rules of differentiation.

The integral of an exponential function with base e is given by the formula:

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C, \quad (\text{for } k \neq 0)$$

This follows from the reverse chain rule: the integral of the exponential part remains unchanged, and we divide by the derivative of the inner function (kx).

For a general exponential function with a base a, where $a > 0$ and $a \neq 1$, the rule is:

$$\int a^{kx}\,dx = \frac{1}{k\ln a}a^{kx} + C, \quad (\text{for } k \neq 0)$$

Example 4.3 Integration of Exponential Functions

Problem: Find the indefinite integral of the function:

$$\int \left(e^{2x} - 5e^{x} + 7\right)\,dx$$

Solution:

Step 1: Recall the standard integral for exponential functions: $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$.

Step 2: Apply this rule to integrate the function term-by-term.

$$\int \left(e^{2x} - 5e^{x} + 7\right)\,dx = \int e^{2x}\,dx - 5\int e^{x}\,dx + \int 7\,dx$$

Step 3: Evaluate each integral and add the constant of integration.

$$= \left(\frac{1}{2}e^{2x}\right) - 5(e^{x}) + 7x + C$$

$$\boxed{\int \left(e^{2x} - 5e^{x} + 7\right)\,dx = \frac{1}{2}e^{2x} - 5e^{x} + 7x + C}$$

■

Q1. Find the indefinite integral of e^{4x+1}. $\left[\textbf{ans: } \frac{1}{4}e^{4x+1} + C\right]$

Q2. Given that a is a positive constant and $\int_{\ln 1}^{\ln a} \left(e^{x} + e^{-x}\right)\,dx = \frac{48}{7}$, find the exact value of a. [**ans:** $a = 7$]

Q3. Find the indefinite integral of $\frac{e^{2x}}{e^{2x}+1}$. $\left[\textbf{ans: } \frac{1}{2}\ln(e^{2x}+1) + C\right]$

Q4. Find the indefinite integral of $\frac{e^{2x}}{(e^{2x}+1)^{3}}$. $\left[\textbf{ans: } -\frac{1}{4(e^{2x}+1)^{2}} + C\right]$

4.1.4 Inverse Trigonometric Functions

Certain algebraic fractions, particularly those involving square roots of quadratics or sums of squares, do not integrate to simple polynomials or logarithms. Instead, their integrals are inverse trigonometric functions. Recognising these standard forms is the key to solving them. The three primary standard forms are listed below. In these formulae, the constant $a > 0$.

- $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C \quad (|x| < a)$
- $\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$
- $\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\operatorname{arcsec}\left(\frac{|x|}{a}\right) + C \quad (|x| > a)$

Example 4.4 Integration involving Inverse Trigonometric Functions

Problem: Find the indefinite integral of the function:

$$\int \left(\frac{1}{1+x^2} + \frac{2}{x^2+4} + \frac{3}{\sqrt{1-x^2}} \right) dx$$

Solution:

Step 1: Recall the standard integrals that result in inverse trigonometric functions.

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right), \quad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right)$$

Step 2: Apply these rules to integrate the function term-by-term.

$$\int \left(\frac{1}{1^2+x^2} + \frac{2}{2^2+x^2} + \frac{3}{\sqrt{1^2-x^2}} \right) dx$$
$$= \int \frac{1}{1+x^2}\,dx + 2\int \frac{1}{x^2+4}\,dx + 3\int \frac{1}{\sqrt{1-x^2}}\,dx$$

Step 3: Evaluate each integral and add the constant of integration.

$$= (\arctan x) + 2\left(\frac{1}{2}\arctan\frac{x}{2}\right) + 3(\arcsin x) + C$$

Step 4: Simplify the expression.

$$= \arctan x + \arctan\frac{x}{2} + 3\arcsin x + C$$

$$\boxed{\int \left(\frac{1}{1+x^2} + \frac{2}{x^2+4} + \frac{3}{\sqrt{1-x^2}} \right) dx = \arctan x + \arctan\frac{x}{2} + 3\arcsin x + C}$$

■

Q1. Integrate the following expressions:

(a) $\displaystyle\int \frac{dx}{\sqrt{1-4x^2}}$ $\left[\textbf{ans: } \frac{1}{2}\arcsin(2x)+C\right]$

(b) $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$ $\left[\textbf{ans: } \arcsin\left(\frac{x}{3}\right)+C\right]$

(c) $\displaystyle\int \frac{12}{1+9x^2}\,dx$ $[\textbf{ans: } 4\arctan(3x)+C]$

(d) $\displaystyle\int \frac{dx}{\sqrt{1-(x+1)^2}}$ $[\textbf{ans: } \arcsin(x+1)+C]$

(e) $\displaystyle\int \frac{dx}{4+(x-3)^2}$ $\left[\textbf{ans: } \frac{1}{2}\arctan\left(\frac{x-3}{2}\right)+C\right]$

(f) $\displaystyle\int \frac{t\,dt}{\sqrt{1-t^4}}$ $\left[\textbf{ans: } \frac{1}{2}\arcsin(t^2)+C\right]$

Q2. Integrate the following expressions:

(a) $\displaystyle\int \frac{t}{t^4+25}\,dt$ $\left[\textbf{ans: } \frac{1}{10}\arctan\left(\frac{t^2}{5}\right)+C\right]$

(b) $\displaystyle\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$ $[\textbf{ans: } \arcsin(\ln x)+C]$

(c) $\displaystyle\int \frac{e^{2x}}{4+e^{4x}}\,dx$ $\left[\textbf{ans: } \frac{1}{4}\arctan\left(\frac{e^{2x}}{2}\right)+C\right]$

(d) $\displaystyle\int \frac{\sec^2 x}{\sqrt{25-\tan^2 x}}\,dx$ $\left[\textbf{ans: } \arcsin\left(\frac{\tan x}{5}\right)+C\right]$

(e) $\displaystyle\int \frac{\sin x}{7+\cos^2 x}\,dx$ $\left[\textbf{ans: } -\frac{1}{\sqrt{7}}\arctan\left(\frac{\cos x}{\sqrt{7}}\right)+C\right]$

(f) $\displaystyle\int \frac{dx}{\sqrt{x(1-x)}}$ $[\textbf{ans: } \arcsin(2x-1)+C]$

Q3. Find the following integrals by splitting them into simpler parts:

(a) $\displaystyle\int \frac{3\,dy}{\sqrt{y(1+y)}}$ $[\textbf{ans: } 3\,\text{arccosh}(2y+1)+C]$

(b) $\displaystyle\int \frac{x-3}{x^2+1}\,dx$ $\left[\textbf{ans: } \frac{1}{2}\ln(x^2+1)-3\arctan(x)+C\right]$

(c) $\displaystyle\int \frac{x+5}{\sqrt{9-(x-3)^2}}\,dx$ $\left[\textbf{ans: } -\sqrt{9-(x-3)^2}+8\arcsin\left(\frac{x-3}{3}\right)+C\right]$

(d) $\displaystyle\int \frac{x-2}{(x+1)^2+4}\,dx$ $\left[\textbf{ans: } \frac{1}{2}\ln((x+1)^2+4)-\frac{3}{2}\arctan\left(\frac{x+1}{2}\right)+C\right]$

4.2 Integration Techniques

4.2.1 Integration by Parts

Integration by parts is a powerful technique used for integrating the product of two functions. It is the integral version of the product rule for differentiation. If u and v are functions of x, the product rule is $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$. By integrating both sides with respect to x and rearranging, we get the formula for integration by parts:

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

This is often written in the more compact form:

$$\int u\,dv = uv - \int v\,du$$

The key to this method is to choose u and dv correctly. The aim is to select a function for u that becomes simpler when differentiated, and a function for dv that can be readily integrated. A helpful guideline for choosing the function for u is the **LIATE** order, which prioritises functions in the following order:

- **L**ogarithmic functions (e.g., $\ln x$)
- **I**nverse trigonometric functions (e.g., $\arcsin x$)
- **A**lgebraic functions (e.g., $x^2, 3x+1$)
- **T**rigonometric functions (e.g., $\sin x, \cos x$)
- **E**xponential functions (e.g., e^x)

Example 4.5 Integration by Parts

Problem: Find the indefinite integral:

$$\int xe^x\,dx$$

Solution:

Step 1: Recall the formula for integration by parts: $\int u\,dv = uv - \int v\,du$.

Step 2: Choose parts for u (the part to differentiate) and dv (the part to integrate).

$$\text{Let } u = x \implies du = dx$$

$$\text{Let } dv = e^x\,dx \implies v = \int e^x\,dx = e^x$$

Step 3: Substitute these parts into the formula.

$$\int xe^x\,dx = (x)(e^x) - \int (e^x)(dx)$$

Step 4: Evaluate the remaining integral and simplify.

$$= xe^x - e^x + C$$

$$= (x-1)e^x + C$$

$$\boxed{\int xe^x\,dx = (x-1)e^x + C}$$

■

Q1. Find the following integrals using integration by parts:

(a) $\int xe^{2x}\,dx$ $\left[\textbf{ans: } \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C\right]$

(b) $\int 3x\cos(2x)\,dx$ $\left[\textbf{ans: } \frac{3}{2}x\sin(2x) + \frac{3}{4}\cos(2x) + C\right]$

(c) $\int x\sin 4x\,dx$ $\left[\textbf{ans: } -\frac{1}{4}x\cos 4x + \frac{1}{16}\sin 4x + C\right]$

(d) $\int -2x\sin 5x\,dx$ $\left[\textbf{ans: } \frac{2}{5}x\cos 5x - \frac{2}{25}\sin 5x + C\right]$

(e) $\int (1-2x)e^{-x}\,dx$ $[\textbf{ans: } (2x+1)e^{-x} + C]$

(f) $\int x^2 e^{-3x}\,dx$ $\left[\textbf{ans: } -\frac{1}{3}x^2e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + C\right]$

(g) $\int 16x^3\ln x\,dx$ $[\textbf{ans: } 4x^4\ln x - x^4 + C]$

(h) $\int \ln x\,dx$ $[\textbf{ans: } x\ln x - x + C]$

4.2.2 Integration by Substitution

Integration by substitution, also known as *u-substitution*, is a technique for reversing the chain rule. The aim is to simplify an integral by changing the variable of integration from x to a new variable, u.

The procedure is to choose a substitution for an "inner function", such as $u = g(x)$, and find its differential, $du = g'(x)dx$. The integral is then rewritten and solved entirely in terms of u. The final step is to back-substitute $u = g(x)$ to express the result in terms of the original variable, x. Formally, this is expressed as:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Example 4.6 Integration by Substitution

Problem: Find the indefinite integral:

$$\int 2x\cos(x^2)\,dx$$

Solution:

Step 1: Choose a substitution for the "inner" part of the function.
Let $u = x^2$.

Step 2: Differentiate the substitution to find du in terms of dx.

$$\frac{du}{dx} = 2x$$
$$\implies du = 2x\,dx$$

Step 3: Rewrite the integral entirely in terms of u and du.
The integral $\int \cos(x^2)\cdot(2x\,dx)$ becomes:

$$\int \cos(u)\,du$$

Step 4: Evaluate the new integral and then back-substitute for u.

$$\int \cos(u)\,du = \sin(u) + C$$
$$= \sin(x^2) + C$$

$$\boxed{\int 2x\cos(x^2)\,dx = \sin(x^2) + C}$$

■

Q1. By using a suitable substitution, find an exact simplified value for:

$$\int_0^1 \frac{10x^4}{2x^5+1}\,dx.$$

[**ans:** $\ln 3$]

Q2. By using the substitution $u = 1 - x^2$, or otherwise, find:

$$\int \frac{12x}{(1-x^2)^{\frac{3}{2}}}\,dx.$$

$\left[\textbf{ans: } \frac{12}{\sqrt{1-x^2}} + C\right]$

Q3. By using the substitution $u^2 = 16 - 7x^2$, or otherwise, show that:

$$\int_0^1 \frac{x}{\sqrt{16-7x^2}}\,dx = \frac{1}{7}.$$

Q4. By using the substitution $u = 1 + 2\cos x$, evaluate the integral:

$$\int_0^{\frac{\pi}{2}} (1+2\cos x)^3 \sin x\,dx.$$

[**ans:** 10]

Q5. By using the substitution $u = 3x+1$, or otherwise, find the exact value of:

$$\int_0^5 \frac{x}{\sqrt{3x+1}}\,dx.$$

[**ans:** 4]

Q6. By using the substitution $u = \sec x$, or otherwise, find:

$$\int \tan x \sec^4 x\,dx.$$

$$\left[\textbf{ans: } \frac{1}{4}\sec^4 x + C\right]$$

4.2.3 Integration by Reverse Chain Rule

The reverse chain rule is not a new rule, but rather a quick method for solving certain integrals by inspection, saving the need to write out a full substitution. It is most effective when the integrand is a product of a composite function and the derivative of its 'inner' function (or a constant multiple of it). This technique applies when you can recognise an integrand in the following form:

$$\int f'(g(x))\,g'(x)\,dx = f(g(x)) + C$$

In simple terms, if you spot an inner function $g(x)$ and its exact derivative $g'(x)$ as a factor, the answer is simply the integral of the outer function f'.

A very common application is when the inner function is linear (of the form $ax+b$). In this case, its derivative is just the constant a. The rule simplifies to:

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C$$

Here, F is the antiderivative of f. The thought process for this shortcut is:

1. Integrate the outer function f, keeping the inner function $(ax+b)$ unchanged.
2. Divide the result by the derivative of the inner function (which is simply the constant a).

Example 4.7 Integration by Reverse Chain Rule

Problem: Find the indefinite integral:

$$\int \cos(5x+2)\,dx$$

Solution:

Step 1: Identify the inner and outer functions.
The outer function is $\cos(u)$ and the inner function is $u = 5x+2$.

Step 2: Integrate the outer function.
The integral of $\cos(u)$ is $\sin(u)$.
$\rightarrow \sin(5x+2)$

Step 3: Differentiate the inner function and divide by the result.
The derivative of the inner function, $5x+2$, is 5.
We divide our result from Step 2 by this derivative.

$$\frac{1}{5}\sin(5x+2)+C$$

$$\boxed{\int \cos(5x+2)\,dx = \frac{1}{5}\sin(5x+2)+C}$$

■

Q1. Find the following integrals using the reverse chain rule (substitution):

(a) $\displaystyle\int (2x+1)\sin\left(x^2+x+1\right)\,dx$ $\quad\left[\textbf{ans: } -\cos(x^2+x+1)+C\right]$

(b) $\displaystyle\int (x+1)\cos\left(x^2+2x+1\right)\,dx$ $\quad\left[\textbf{ans: } \frac{1}{2}\sin(x^2+2x+1)+C\right]$

(c) $\displaystyle\int \frac{1}{x(1+\ln x)^3}\,dx$ $\quad\left[\textbf{ans: } -\frac{1}{2(1+\ln x)^2}+C\right]$

(d) $\displaystyle\int \left(4-\cos^4 x\sin x\right)\,dx$ $\quad\left[\textbf{ans: } 4x+\frac{1}{5}\cos^5 x+C\right]$

(e) $\displaystyle\int \frac{\cos x}{\sin^3 x}\,dx$ $\quad\left[\textbf{ans: } -\frac{1}{2}\csc^2 x+C\right]$

(f) $\displaystyle\int \frac{\sqrt{1+2\tan x}}{\cos^2 x}\,dx$ $\quad\left[\textbf{ans: } \frac{1}{3}(1+2\tan x)^{\frac{3}{2}}+C\right]$

(g) $\displaystyle\int \frac{\cos x}{\sqrt{\sin x}}\,dx$ $\quad\left[\textbf{ans: } 2\sqrt{\sin x}+C\right]$

4.3 Multiple Integration

Multiple integration extends the concept of a standard integral to functions of more than one variable. This allows us to calculate quantities such as area, volume, and mass over two or three-dimensional regions.

Double Integrals

A double integral is used to integrate a function of two variables, $f(x,y)$, over a region R in the xy-plane. If $f(x,y) \geq 0$, the double integral geometrically represents the volume of the solid that lies above the region R and below the surface defined by $z = f(x,y)$.

$$\text{Volume} = \iint_R f(x,y)\,dA$$

The term dA represents an infinitesimal area element in the plane, which is typically written as $dx\,dy$ or $dy\,dx$.

Evaluating Double Integrals as Iterated Integrals

We evaluate double integrals using a method called **iterated integration**, which means performing single-variable integrations one after the other. When integrating with respect to one variable, all other variables are treated as constants. For a rectangular region R defined by $a \leq x \leq b$ and $c \leq y \leq d$, the double integral is calculated as:

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

For rectangular regions, the order of integration ($dx\,dy$ or $dy\,dx$) can be swapped without changing the final result.

Triple Integrals

The same principle extends to three dimensions. A triple integral evaluates a function $f(x,y,z)$ over a 3D region V. While a double integral can represent volume, a triple integral can be used to find quantities such as the mass of an object with a variable density function $\rho(x,y,z)$:

$$\text{Mass} = \iiint_V \rho(x,y,z)\,dV$$

These are also calculated as three successive iterated integrals.

Example 4.8 Evaluating a Double Integral over a Rectangle

Problem: Evaluate the double integral:

$$\int_0^1 \int_0^2 (x+2y)\,dy\,dx$$

Solution:

Step 1: Recall the process for evaluating an iterated integral.

We evaluate the integral from the inside out, starting with dy.

$$\int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

Step 2: Evaluate the inner integral with respect to y, treating x as a constant.

$$\int_0^2 (x+2y)\,dy = \left[xy+y^2\right]_{y=0}^{y=2}$$
$$= \left(x(2)+(2)^2\right) - \left(x(0)+(0)^2\right)$$
$$= 2x+4$$

Step 3: Substitute this result into the outer integral and evaluate with respect to x.

$$\int_0^1 (2x+4)\,dx = \left[x^2+4x\right]_{x=0}^{x=1}$$
$$= \left((1)^2+4(1)\right) - \left((0)^2+4(0)\right)$$
$$= 5$$

$$\boxed{\int_0^1 \int_0^2 (x+2y)\,dy\,dx = 5}$$

■

Q1. Find the following indefinite iterated integrals:

(a) $\iint xy\sqrt{1+x^2+y^2}\,dx\,dy$

$$\left[\textbf{ans: } \frac{1}{15}(1+x^2+y^2)^{5/2} + g(x) + h(y)\right]$$

(b) $\iint \frac{1}{(x+y+1)^3}\,dx\,dy$

$$\left[\textbf{ans: } \frac{1}{2(x+y+1)} + g(x) + h(y)\right]$$

(c) $\iint x\sin(xy)\,dy\,dx$

$$\left[\textbf{ans: } -\frac{\sin(xy)}{y} + g(x) + h(y)\right]$$

(d) $\iint (2x - 3y^2)\,dx\,dy$

$\left[\textbf{ans: } x^2y - xy^3 + g(x) + h(y)\right]$

(e) $\iint x\cos(x^2 + y)\,dx\,dy$

$\left[\textbf{ans: } -\frac{1}{2}\cos(x^2 + y) + g(x) + h(y)\right]$

Q2. Evaluate the following definite integrals:

(a) $\iint_D (42y^2 - 12x)\,dA$ where $D = \{(x,y) \mid 0 \leq x \leq 4, (x-2)^2 \leq y \leq 6\}$.

[**ans:** 11136]

(b) $\iint_D (2yx^2 + 9y^3)\,dA$ where D is the region bounded by $y = \frac{2}{3}x$ and $y = 2\sqrt{x}$.

$\left[\textbf{ans: } \frac{24057}{5}\right]$

(c) $\iint_D (10x^2y^3 - 6)\,dA$ where D is the region bounded by $x = -2y^2$ and $x = y^3$.

$\left[\textbf{ans: } -\frac{8296}{13}\right]$

(d) $\iint_D x(y-1)\,dA$ where D is the region bounded by $y = 1 - x^2$ and $y = x^2 - 3$.

[**ans:** 0]

(e) $\iint_D 5x^3\cos(y^3)\,dA$ where D is the region bounded by $y = 2$, $y = \frac{1}{4}x^2$, and the y-axis.

$\left[\textbf{ans: } \frac{20}{3}\sin(8)\right]$

4.4 Chapter Review

Review Exercise 7 Mixed Questions

Problem 4.4.1 Evaluate the definite integral by using a suitable substitution:

$$\int_0^{\pi/2} e^{\sin(x)} \cos(x)\, dx$$

Problem 4.4.2 Find the indefinite integral by applying integration by parts twice:

$$\int x^2 \sin(x)\, dx$$

Problem 4.4.3 Find the indefinite integral by first splitting the fraction into simpler parts:

$$\int \frac{2x+3}{x^2+9}\, dx$$

Problem 4.4.4 Evaluate the double integral $\iint_D (x+y)\, dA$, where D is the triangular region in the xy-plane bounded by the lines $y=0$, $x=1$, and $y=x$.

Problem 4.4.5 Find the indefinite integral by first using a trigonometric identity:

$$\int \sin^3(x)\, dx$$

Problem 4.4.6 Find the exact value of the definite integral using integration by parts:

$$\int_1^e x \ln(x)\, dx$$

Problem 4.4.7 Evaluate the integral by making a suitable trigonometric substitution.

$$\int_0^3 \sqrt{9-x^2}\, dx$$

Problem 4.4.8 Evaluate the double integral by first reversing the order of integration:

$$\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$$

Problem 4.4.9 Find the indefinite integral using partial fraction decomposition:

$$\int \frac{x+1}{x(x-1)^2}\, dx$$

■

Summary 4 Integration

Basic Rules:

- **Constant of Integration:** Indefinite integrals must include a constant, $+C$.
- **Power Rule:** $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ (for $n \neq -1$)
- **Sum/Difference:** $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
- **Constant Multiple:** $\int kf(x)\,dx = k\int f(x)\,dx$

Key Integration Techniques:

- **By Substitution (Reverse Chain Rule):** Used to simplify integrals by changing the variable. $\int f(g(x))g'(x)\,dx = \int f(u)\,du$.
- **By Parts (Reverse Product Rule):** Used to integrate products of functions. $\int u\,dv = uv - \int v\,du$.

Integrals of Standard Functions:

- **Exponential:** $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$
- **Reciprocal:** $\int \frac{1}{x}\,dx = \ln|x| + C$
- **Trigonometric:** $\int \cos(x)\,dx = \sin(x) + C, \quad \int \sin(x)\,dx = -\cos(x) + C$
- **Inverse Trig:** $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$

Chapter Checklist

- ☐ I can find indefinite integrals using the power rule and linearity.
- ☐ I can integrate standard exponential, logarithmic, and trigonometric functions.
- ☐ I can use integration by substitution to solve integrals.
- ☐ I can use integration by parts to solve integrals.
- ☐ I can recognise integrals that lead to inverse trigonometric forms.
- ☐ I can evaluate definite integrals.
- ☐ I can set up and evaluate a double integral over a given region.

■