INSTITUTE OF
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Differentiation – Tutor Notes

This session builds on the differentiation concepts introduced in the prereading, providing additional examples and explanations to reinforce understanding. Begin by assessing students' current knowledge – ask what they recall about differentiation from A-level or from the prereading. Use this to identify which sections may need clarification. If students express uncertainty about a particular topic, or indicate they found a section challenging, you can navigate directly to the relevant guidance below. The material is structured to follow the prereading sequence, allowing you to address specific gaps based on your student cohort's confidence levels.

Part I

Welcome Back – 15 Minutes

The derivative definition: Ask your students "What does $\frac{dy}{dx}$ actually mean?" Listen for answers like "rate of change" or "slope of the tangent". Refer them to the figure in the prereading and walk them through the four key aspects:

1. The curve $y = f(x)$ – this is our function
2. The point P – where we want the gradient
3. The tangent line – the line that just touches at P
4. The gradient of this tangent – this is $\frac{dy}{dx}$ at P

Reinforce this throughout: the derivative is fundamentally the gradient of the tangent at a point. Get them to visualise this before doing any algebra.

Differentiation rules: Work through these examples on a shared whiteboard, sketching each one:

a) Start with $y = 15$. Sketch the horizontal line and ask "What's the gradient?" Hence, $\frac{dy}{dx} = 0$.
b) Move to $y = 2x$. Sketch the line through the origin. Remind them of $y = mx + c$ form – gradient is constant at 2, so $\frac{dy}{dx} = 2$.
c) Now try $y = -x^2$. Sketch the sad parabola. Ask "Is the gradient the same everywhere?" They should see it changes with x. Differentiate to show $\frac{dy}{dx} = -2x$ – point out this is now a *function* of x, unlike the previous examples. This is a key insight.
d) Finish with $y = x^2 + 5$. Sketch the happy parabola shifted up. Differentiate to get $\frac{dy}{dx} = 2x$ – note how the constant $+5$ disappears. Ask them why (derivative of constant is zero).

Stationary points: Using your sketch of $y = x^2 + 5$, show how the gradient goes from negative → zero → positive through the lowest point. Explain that this zero-gradient point is called a **stationary point** (or turning point). Ask them: "How could we find this point without drawing the graph?" Guide them to setting $\frac{dy}{dx} = 0$. For $y = x^2 + 5$, $2x = 0 \Rightarrow x = 0$.

Classifying maxima/minima: Refer to Section 5 of the prereading. Explain the second derivative test simply:

- If $f''(x) > 0$, the graph is concave up (like a cup) → minimum
- If $f''(x) < 0$, the graph is concave down (like a cap) → maximum

For $y = x^2 + 5$, $f''(x) = 2 > 0$ confirms a minimum at $x = 0$. Get them to check this against their sketch.

Kinematics: Tell your students this is directly tested in A-level Mechanics, so it's worth mastering. Walk them through the derivative journey:

$$s(t) \xrightarrow{\frac{d}{dt}} v(t) = \frac{ds}{dt} \xrightarrow{\frac{d}{dt}} a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Emphasise the units: metres (m) → metres per second (m/s) → metres per second squared (m/s^2).

Then introduce angular motion (Section 6 of prereading):

$$\theta(t) \xrightarrow{\frac{d}{dt}} \omega(t) = \frac{d\theta}{dt} \xrightarrow{\frac{d}{dt}} \alpha(t) = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

Units: radians (rad) → rad/s → rad/s^2.

Worked example: Take them through a cubic angular function:

$$\theta(t) = t^3 - 6t^2 + 9t$$

Differentiate once: $\omega(t) = 3t^2 - 12t + 9$ (quadratic). Differentiate again: $\alpha(t) = 6t - 12$ (linear). Point out the pattern – each differentiation reduces the polynomial degree by one. This is a handy check.

Trigonometric and exponential functions: Refer to Section 7 of the prereading to show:

$$\frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x, \quad \frac{d}{dx}e^{kx} = ke^{kx}$$

Explain that these are everywhere in engineering – oscillations, AC circuits, capacitor charging. They're essential tools, so commit them to memory.

Advanced rules: Let your students know they'll encounter product, quotient, and chain rules in Questions 9-10. These are covered in Section 8 of the prereading. Reassure them that they don't need to memorise everything immediately – they can refer back to the prereading as needed. The key is recognising which rule applies in a given situation, not rote memorisation.

Part II

Getting Started – 10 Minutes

Let students work on **Questions 1 and 2** and then walk through the answers together.

- **Q1:** This is straightforward power rule practice. Check that students included units (ohms) and understood that instantaneous resistance is the derivative – it's not V/I, which would give average resistance. This is a common confusion worth addressing.
- **Q2:** Verify students formed the correct area function $A(x) = 16x - 2x^2$, found $A'(x) = 16 - 4x$, and set it to zero to get $x = 4$. For part (c), show them there are two ways to verify it's a maximum – either use the second derivative test ($A''(x) = -4 < 0$) or simply recognise that $A(x)$ is a concave-down parabola (coefficient of x^2 is negative), so its turning point must be a maximum. Both approaches reinforce different aspects of the prereading.

Getting Stuck In – 30 Minutes

Ask students to focus on **Questions 3 to 5** in that order. Pay special attention to:

- **Q3:** This question is great for reinforcing physical interpretation. When they find $\frac{dy}{dF} = 0.04F + 0.3$, ask them "What does this actually mean?" Guide them to see it's the extra deflection per additional kN of load. The positive second derivative ($0.04 > 0$) tells them the arm becomes less stiff as load increases – each extra kN causes more deflection than the last.
- **Q4:** This question ties together many concepts. Work through it carefully:
  - Part (a) is straightforward power rule application – but point out how the quartic reduces to cubic (velocity) then quadratic (acceleration). This pattern reinforces earlier discussion.

  - In part (b), ensure students don't miss the $t = 0$ solution when solving $\omega(t) = 0$. It's easy to overlook!
  - Part (c) often challenges intuition. The idea of a flywheel momentarily stopping then reversing direction isn't obvious. Show them how checking the sign of ω just before and after each root tells us what's happening. For $t \approx 5.56$, $\omega(5) > 0$ and $\omega(6) < 0$ – that sign change means reversal. For $t = 0$, we only have values after, so it's starting from rest – no reversal.
  - Throughout parts (d) and (e), reinforce the link: when $\alpha(t) = 0$, $\omega(t)$ has a stationary point. This mirrors the displacement-velocity relationship they already know.
- **Q5:** This conical tank problem introduces related rates – a new application. The key step many miss is using the chain rule: $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$. Walk them through this carefully. Also, part (e) is worth discussing – as $h \to 0$, $\left|\frac{dh}{dt}\right| \to \infty$. Ask them "What does this mean for the engineer designing the drainage system?"

Break

Encourage students to step away from screens briefly.

Part III

More Problem Solving – 30 Minutes

Ask students to focus on **Questions 6 to 8**, then go through the solutions together.

- **Q6:** This is a short but important question linking differentiation to physics. Refer them back to Section 7 of the prereading and show:

$$\frac{d}{dt}\cos t = -\sin t = -E(t), \quad \frac{d}{dt}\sin t = \cos t = B(t)$$

- **Q7:** This question anchors back to A-level linear kinematics. Take time to stress the vertical/horizontal split:
  - Only vertical motion has a force (weight) – horizontal motion has no force, hence constant velocity. This is why the x-equation is linear and the y-equation is quadratic.
  - Vertically, initial velocity is upward, decreases to zero at maximum height, then increases downward.
  - Ask students: "What is the velocity when the object hits the ground?" If anyone says 0 m/s, correct this misconception! It hits the ground with non-zero velocity (same magnitude as launch speed but downward direction). The SUVAT equations confirm this.
  - Part (f) involves solving a quadratic inequality – remind them the two roots are the times when the object passes 2 m on the way up and on the way down.
- **Q8:** This question introduces exponential differentiation. Walk through carefully:
  - Part (b): $I(t) = C\frac{dV}{dt}$. The differentiation $\frac{d}{dt}(1 - e^{-t/20}) = \frac{1}{20}e^{-t/20}$ can trip students up. Show the chain rule explicitly.
  - Parts (b) and (d) involve fractions – encourage neat, step-by-step working to avoid errors.
  - Check units consistently: I in amps, $\frac{dI}{dt}$ in A/s. Ask students why part (d) has units A/s – it's rate of change of current, so current per second.
  - The time constant $\tau = RC = 20$ s is a key concept – highlight that after one time constant, the current has dropped to $1/e$ of its initial value (about 37%).

Wrap Up – 5 Minutes

Use the final few minutes to signpost remaining questions as extensions for students to attempt in their own time:

- **Q9:** This is a heat transfer application that brings together several concepts. Students will need the product rule from Section 8 of the prereading to handle part (d), where thermal conductivity depends on temperature. Encourage them to work through it step by step – the physics might be new, but the differentiation is manageable.

- **Q10:** This question provides practice with the quotient rule (also Section 8). The key insight in part (c) is that finding where stress increases most rapidly means maximising $\frac{d\sigma}{dr}$ – which requires differentiating again. This is a common engineering task: identifying regions of high sensitivity. In part (d), ask them to think about why engineers care about these regions – small manufacturing tolerances can lead to large stress variations.

Remind students that attempting these in their own time consolidates learning, but they should only refer to the solutions after giving the problems a genuine try. Leave them with the message that differentiation is a toolkit – the more they practice, the more naturally they'll recognise which tool to use.