# Differentiation – Solutions

1. In an electrical circuit test for a power supply, the voltage drop $V$ (in volts) across a component varies with current $I$ (in amps) is $V = 10I+0.5I^2$. For this non-linear component, the instantaneous resistance ($R$) is defined as the slope of the $V - I$ curve. Find $R$ when the current through the component is 0.1 amps.

## Solution:

The voltage drop $V$ (in volts) is given as a function of current $I$ (in amps):

$$V = 10I + 0.5I^2.$$

The instantaneous resistance $R$ is defined as the slope of the $V$–$I$ curve, i.e. the derivative of $V$ with respect to $I$:

$$R = \frac{dV}{dI}.$$

Differentiate term by term:

$$\frac{dV}{dI} = \frac{d}{dI}(10I) + \frac{d}{dI}(0.5I^2).$$

Apply the power rule $\frac{d}{dI}(I^n) = nI^{n-1}$:

$$\frac{dV}{dI} = 10 \cdot 1 + 0.5 \cdot 2I = 10 + I.$$

Now substitute the given current $I = 0.1$ amps:

$$R = 10 + 0.1 = 10.1\ \Omega.$$

Therefore, the instantaneous resistance at 0.1 amps is $\boxed{10.1\ \Omega}$.

2. A chemical engineer is designing a rectangular reaction tank next to a factory wall, with one long side running along the wall (no lining required there). The total length of lining material available for the remaining three sides is 16m. Let the width perpendicular to the wall be $x$ m.
   (a) Express the tank's base area $A$ in terms of $x$ only.
   (b) Differentiate $A$ with respect to $x$ and find the dimensions that maximise the area.
   (c) By considering the second derivative or otherwise, verify that your dimensions give a maximum.
   (d) Sketch the graph of $A$ against $x$. State the range of values for $x$ and verify that your maximum lies within it.

## Solution:

Let width perpendicular to wall be $x$ m, length parallel to wall be $L$ m. Lining available for three sides: $2x + L = 16 \Rightarrow L = 16 - 2x$.

**(a) Base area $A$ in terms of $x$**
The base area is $A = \text{length} \times \text{width} = L \cdot x$.

$$A(x) = (16 - 2x)x = 16x - 2x^2.$$

**(b) Differentiate $A$ and find dimensions for maximum area**

$$\frac{dA}{dx} = 16 - 4x.$$

Set derivative to zero for stationary point:

$$16 - 4x = 0 \implies x = 4.$$

Then $L = 16 - 2(4) = 8$. Thus the dimensions are:

$$\boxed{x = 4 \text{ m}, \qquad L = 8 \text{ m}}$$

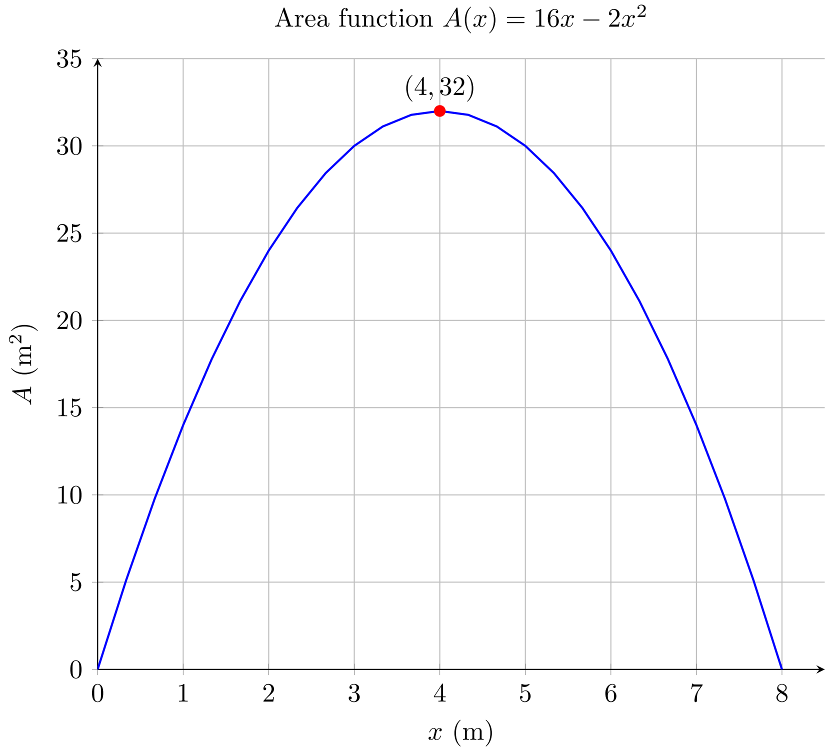
Maximum area: $A_{\text{max}} = 4 \times 8 = 32 \text{ m}^2$.

**(c) Verification of maximum**
Second derivative:

$$\frac{d^2A}{dx^2} = -4.$$

Since $\frac{d^2A}{dx^2} < 0$, the stationary point is a maximum.
This is also clear from the fact that $A(x) = 16x - 2x^2$ is a negative (concave-down) parabola, so its turning point must be a maximum.

**(d) Sketch of $A$ against $x$ and feasible range**
The function is $A(x) = 16x - 2x^2$, a concave-down parabola.




**Range of $x$:** Length $L = 16 - 2x$ must be positive, and width $x$ must be positive:

$$x > 0, \quad 16 - 2x > 0 \implies x < 8.$$

Hence the domain is $0 < x < 8$. At the endpoints $x = 0$ and $x = 8$ the area is zero.
The turning point (maximum) occurs at $x = 4$, which lies inside the interval $(0, 8)$.

3. A rigid arm holds a set of traffic lights out over a junction. Engineers model the vertical deflection y (in mm) of the tip of the arm under load by:

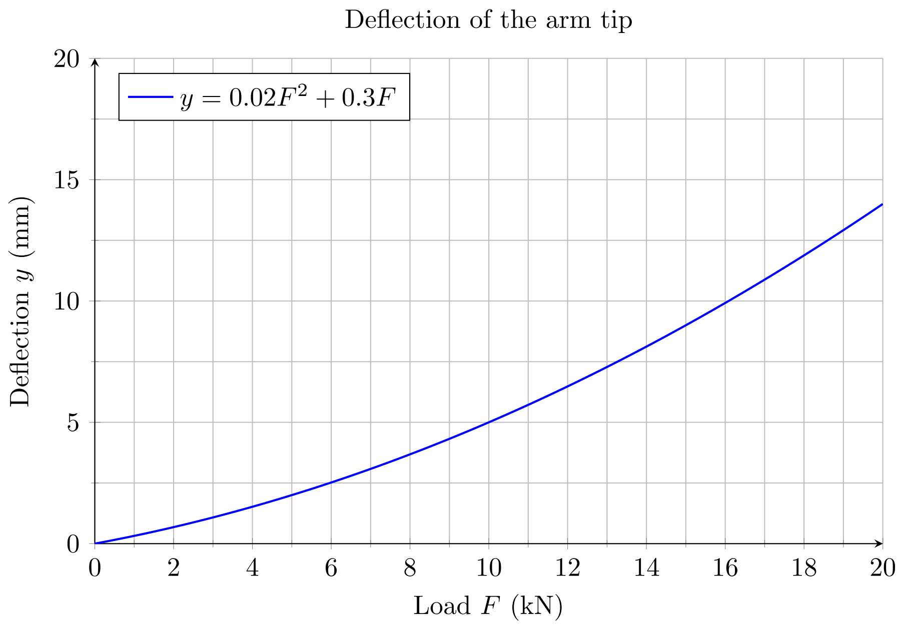
$$y(F) = 0.02F^2 + 0.3F, \quad F \geq 0$$

where $F$ is the total equipment load at the tip in kN (lights, cameras, signage, etc.).

(a) Plot a graph of $y$ against $F$ for $0 \leq F \leq 20$.
(b) Find $\frac{dy}{dF}$, state the units and interpret this derivative physically.
(c) Find $\frac{d^2y}{dF^2}$, state the units and explain what the sign implies about how the arm's stiffness changes as the load due to the lights increases.
(d) At what value of $F$ does an additional 1 kN of load cause the greatest extra deflection on the rigid arm?

## Solution:

### (a) Graph of $y$ against $F$ for $0 \leq F \leq 20$




### (b) First derivative $\dfrac{dy}{dF}$

$$\frac{dy}{dF} = \frac{d}{dF}\big(0.02F^2 + 0.3F\big) = 0.04F + 0.3.$$

- **Units:** $y$ is in mm, $F$ is in kN, so $\dfrac{dy}{dF}$ is in mm/kN.
- **Physical interpretation:** The derivative represents the *instantaneous rate of change* of deflection with respect to load. In other words, it is the additional deflection (in mm) that would be caused by a very small increase in load (1 kN for practical purposes). It is the slope of the deflection–load curve.

### (c) Second derivative $\dfrac{d^2y}{dF^2}$

$$\frac{d^2y}{dF^2} = \frac{d}{dF}(0.04F + 0.3) = 0.04.$$

- **Units:** mm/kN$^2$.
- **Interpretation of the sign:** The second derivative is *positive* ($0.04 > 0$). This means the slope $\frac{dy}{dF}$ increases as $F$ increases. Physically, the arm becomes *less stiff* under larger loads: each additional kN of load causes a larger extra deflection than the previous one. The arm "softens" as the load grows.

### (d) Greatest extra deflection from an additional 1 kN

The greatest extra deflection from an additional 1 kN occurs when $\dfrac{dy}{dF}$ is largest.

$$\frac{dy}{dF} = 0.04F + 0.3.$$

This is a linear, increasing function of $F$. Therefore, the greatest value of $\frac{dy}{dF}$ on the interval $0 \le F \le 20$ occurs at the largest permissible load.

$$\left.\frac{dy}{dF}\right|_{F=20} = 0.04(20) + 0.3 = 1.1 \text{ mm/kN}.$$

Thus, an additional 1kN load causes the greatest extra deflection when the load is at its maximum, i.e. at

$$\boxed{F = 20 \text{ kN}}.$$

(If no upper bound were stated, the function would increase without limit, so no finite maximum exists. Within the plotted range $0 \le F \le 20$, the maximum is at $F = 20$.)

4. In a flywheel energy storage system, a rotating flywheel stores kinetic energy. The angular position (in radians) of the flywheel during a testing cycle is given by:

$$\theta(t) = 0.1t^4 - 2t^3 + 10.5t^2, \quad 0 \le t \le 10$$

where $t$ is time in seconds.

(a) Find the angular velocity $\omega(t)$ and angular acceleration $\alpha(t)$. State the units.
(b) Determine all times $t$ in $[0, 10]$ when the flywheel is momentarily at rest.
(c) For each value of time found in (b), decide whether the flywheel reverses direction or momentarily stops and then continues in the same direction. Explain your reasoning.
(d) Find the time(s) when the angular acceleration is zero. What does this imply about the rotational motion?
(e) Use a sketch or a graphing software plot to illustrate the relationship between the angular displacement, velocity, and acceleration curves. Relate this to your answer in part (d).

## Solution:

**(a) Angular velocity and angular acceleration**

Angular velocity $\omega(t) = \dfrac{d\theta}{dt}$:

$$\omega(t) = \frac{d}{dt}\big(0.1t^4 - 2t^3 + 10.5t^2\big) = 0.4t^3 - 6t^2 + 21t.$$

**Units:** $\omega(t)$ is in rad/s.

Angular acceleration $\alpha(t) = \dfrac{d\omega}{dt}$:

$$\alpha(t) = \frac{d}{dt}\big(0.4t^3 - 6t^2 + 21t\big) = 1.2t^2 - 12t + 21.$$

**Units:** $\alpha(t)$ is in rad/s$^2$.

**(b) Times when the flywheel is momentarily at rest**

The flywheel is at rest when $\omega(t) = 0$:

$$0.4t^3 - 6t^2 + 21t = t\big(0.4t^2 - 6t + 21\big) = 0.$$

Hence either $t = 0$ or $0.4t^2 - 6t + 21 = 0$. Solving the quadratic:

$$t \approx 5.5635, \qquad t \approx 9.4365.$$

All three values lie in $[0, 10]$. Thus the flywheel is momentarily at rest at

$$\boxed{t = 0,\ t \approx 5.56 \text{ s},\ t \approx 9.44 \text{ s}}.$$

**(c) Direction reversal or momentary stop?**

Examine the sign of $\omega(t)$ immediately before and after each zero.

- **At $t = 0$:** $\omega(0.1) \approx 2.04 > 0$. The flywheel starts from rest and immediately moves forward – it does **not** reverse direction.
- **At $t \approx 5.56$:** $\omega(5) = 5 > 0$ and $\omega(6) \approx -3.6 < 0$. The velocity changes from positive to negative, so the flywheel **reverses direction**.
- **At $t \approx 9.44$:** $\omega(9) \approx -5.4 < 0$ and $\omega(10) = 10 > 0$. The velocity changes from negative to positive, again indicating a **reversal of direction**.

**(d) Times when angular acceleration is zero**
Solving $\alpha(t) = 1.2t^2 - 12t + 21 = 0$,

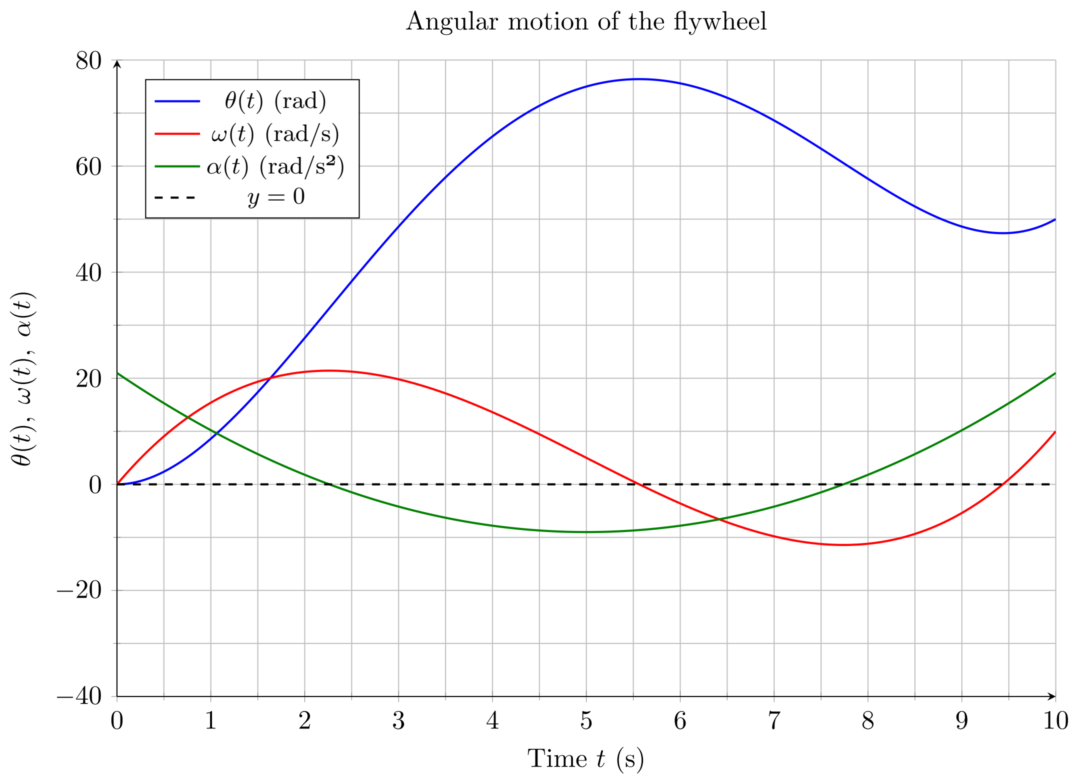
$$t \approx 2.2615, \qquad t \approx 7.7385.$$

Both values lie in $[0, 10]$.

$$\boxed{t \approx 2.26 \text{ s}, \ t \approx 7.74 \text{ s}}$$

**Interpretation:** When $\alpha(t) = 0$, the angular velocity $\omega(t)$ has a horizontal tangent; these are the instants where $\omega$ attains a local maximum or minimum.

**(e) Sketch of displacement, velocity and acceleration curves**




The plot shows the relationship among the three functions. The points where $\alpha(t) = 0$ correspond exactly to the local stationary point of $\omega(t)$:

- At $t \approx 2.26$s, $\omega$ reaches a local maximum.
- At $t \approx 7.74$s, $\omega$ reaches a local minimum.

This confirms that zero acceleration marks turning points in the velocity profile (and not in the displacement directly.)

**Note:** The turning points on the displacement profile correspond to the time instants when the velocity profile intersects the $y = 0$ line (the $x$-axis).

5. Water is draining from a conical tank with vertex angle $60°$. At any time $t$ seconds, let the radius of the surface be $r$ metres and the height of water be $h$ metres. The volume of water in the tank is:

$$V = \frac{1}{3}\pi r^2 h$$

and for this tank, $r = h\tan(30^\circ) = \frac{h}{\sqrt{3}}$.

(a) Show that $V = \frac{\pi}{9}h^3$.

(b) Water drains at a constant rate of 0.02 $\text{m}^3\text{s}^{-1}$. Find the surface drop rate, $\frac{dh}{dt}$, when $h = 1.5\,\text{m}$ and give its units.

(c) Explain the physical meaning of your result from (b) in relation to its sign and magnitude.

(d) Find the depth $h_d$ where the magnitude of the surface drop rate $\left|\frac{dh}{dt}\right|$ is twice the value found in (b).

(e) The tank starts full at $h = 3m$. As $h$ approaches zero, explain whether $\left|\frac{dh}{dt}\right|$ increases or decreases. Interpret what your answer means for the engineer designing this drainage system.

## Solution:

### (a) Volume in terms of $h$.

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h^2}{3}\right)h = \frac{\pi}{9}h^3$$

Thus $\boxed{V = \frac{\pi}{9}h^3}$

### (b) Surface drop rate $\frac{dh}{dt}$ when $h = 1.5$ m

Water drains at a constant rate: $\frac{dV}{dt} = -0.02\ \text{m}^3/\text{s}$ (negative because volume decreases).

Differentiate $V = \frac{\pi}{9}h^3$ with respect to $t$:

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt} = \frac{\pi}{9}\cdot 3h^2\frac{dh}{dt} = \frac{\pi}{3}h^2\frac{dh}{dt}$$

Hence

$$\frac{dh}{dt} = \frac{\frac{dV}{dt}}{\frac{\pi}{3}h^2} = \frac{-0.02}{\frac{\pi}{3}h^2} = -\frac{0.06}{\pi h^2}$$

At $h = 1.5$ m:

$$\frac{dh}{dt} = -\frac{0.06}{\pi(1.5)^2} \approx -0.00849\ \text{m/s}.$$

$$\boxed{\frac{dh}{dt} \approx -0.00849\ \text{m/s}}\quad \left(\text{or } -\frac{0.06}{2.25\pi}\ \text{m/s}\right).$$

### (c) Physical interpretation

The negative sign indicates that the water level is *decreasing* (falling). The magnitude $\left|\frac{dh}{dt}\right| \approx 0.00849$ m/s tells us that when the water depth is 1.5 m, the surface drops at a rate of about 8.5 millimetres per second. This rate is not constant; it depends on the current depth $h$.

### (d) Depth $h_d$ where $\left|\frac{dh}{dt}\right|$ is twice the value at $h = 1.5$ m

From part (b): $\left|\frac{dh}{dt}\right| = \frac{0.06}{\pi h^2}$. At $h = 1.5$: $\left|\frac{dh}{dt}\right|_{h=1.5} = \frac{0.06}{2.25\pi}$.

We require $\frac{0.06}{\pi h_d^2} = 2 \times \frac{0.06}{2.25\pi}$. Solving this for $h_d^2$:

$$\frac{1}{h_d^2} = \frac{2}{2.25} \quad\Rightarrow\quad h_d^2 = \frac{9}{8}.$$

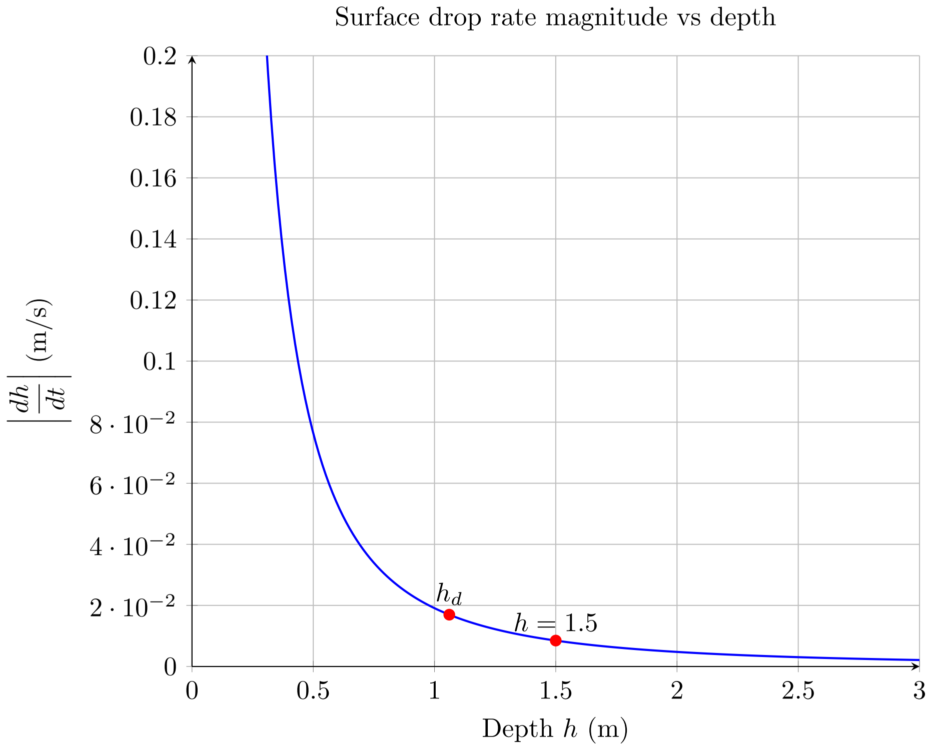
$$\boxed{h_d = \frac{3\sqrt{2}}{4}\ \text{m} \approx 1.061\ \text{m}}.$$

In other words, at a depth of about 1.06 m, the water level falls twice as fast as it did at 1.5 m.

**(e) Behaviour as $h \to 0$; engineering implication**
The magnitude of the surface drop rate is

$$\left|\frac{dh}{dt}\right| = \frac{0.06}{\pi h^2}.$$

As $h \to 0$, $h^2 \to 0$ so $\frac{1}{h^2} \to \infty$; hence $|\frac{dh}{dt}| \to \infty$. This means the water level falls *extremely rapidly* just before the tank empties.




**Engineering interpretation:** The rate at which the water level drops depends inversely on the square of the current height: when the tank is fuller (larger $h$), the level drops more slowly; as the tank empties (smaller $h$), the level drops more rapidly. This non-linear behaviour occurs because the cross-sectional area of the water surface decreases with height in a conical tank. Engineers must account for this when designing drainage systems to ensure that drainage times are appropriate and the outlet pipe or valve can accommodate this rapid drop without cavitation, vortexing, or exceeding the structural limits of the tank bottom.

6. During the testing of an electromagnetic field sensor, the measured electric and magnetic fields are found to be:

$$E(t) = \sin t, \qquad B(t) = \cos t.$$

Maxwell's equations in free space relate these fields by:

$$E = -\frac{dB}{dt} \quad \text{and} \quad B = \frac{dE}{dt}.$$

Verify that the measured fields satisfy both Maxwell's equations.

## Solution:

First, compute $\frac{dB}{dt}$:

$$\frac{dB}{dt} = \frac{d}{dt}(\cos t) = -\sin t = -E(t).$$

Hence $E(t) = -\frac{dB}{dt}$ is satisfied.

Second, compute $\frac{dE}{dt}$:

$$\frac{dE}{dt} = \frac{d}{dt}(\sin t) = \cos t = B(t).$$

Hence $B(t) = \frac{dE}{dt}$ is satisfied.

$$\boxed{\text{Both Maxwell's equations are satisfied.}}$$

**Note:** The sinusoidal electric and magnetic fields are exactly out of phase by 90° and are consistent with electromagnetic wave propagation in free space, where the time rate of change of one field generates the other.

7. A quality control technician launches a small calibration weight from a testing rig to verify it clears a safety barrier during automated assembly. The weight is projected from ground level with an initial speed of $18\,\text{ms}^{-1}$ at 40° above the horizontal. Assume that the acceleration due to gravity $g = 9.8\,\text{ms}^{-2}$.
   (a) State the horizontal and vertical components of the initial velocity.
   (b) Assuming no air resistance, list all forces acting on the calibration weight and their respective directions.
   (c) Express the vertical height $y$ metres at time $t$ seconds. Use a graphing software to show the relationship between $y$ and $t$ and state the polynomial order of this equation.
   (d) Find the horizontal distance from launch at maximum height.
   (e) Find the total horizontal range before it returns to ground.
   (f) For installation planning, engineers need the calibration weight to remain at least 2 m above ground while passing over the safety barrier. Using your model $y(t)$, find how long the component stays above 2 m.

## Solution:

**(a) Horizontal and vertical components of initial velocity**

$$u_x = u\cos\theta = 18\cos 40^\circ, \qquad u_y = u\sin\theta = 18\sin 40^\circ.$$

Numerically:

$$\cos 40^\circ \approx 0.7660, \quad \sin 40^\circ \approx 0.6428,$$

$$u_x \approx 18 \times 0.7660 = 13.788 \text{ m/s}, \qquad u_y \approx 18 \times 0.6428 = 11.570 \text{ m/s}.$$

$$\boxed{u_x \approx 13.79 \text{ m/s}, \ u_y \approx 11.57 \text{ m/s}}.$$

**(b) Forces and their directions**

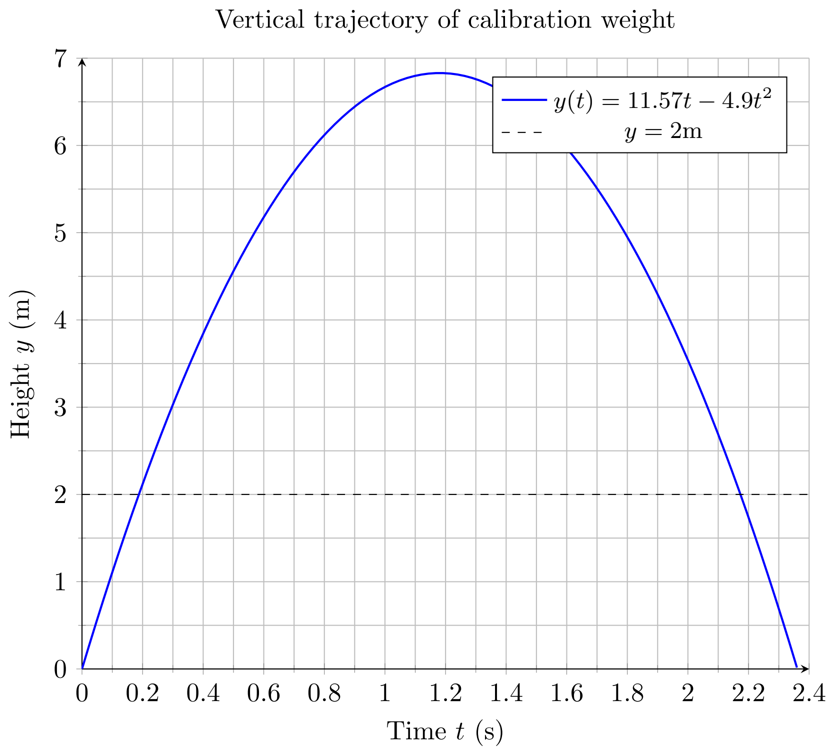
Only one force acts on the calibration object and that is it's weight, acting *vertically downward*. No horizontal force acts on the object.

**(c) Vertical height $y(t)$ and graph**

Using $s = ut + \frac{1}{2}at^2$ with upward positive and $a = -g$:

$$y(t) = u_y t - \frac{1}{2}gt^2 = 11.57t - 4.9t^2 \quad \text{(metres)}.$$

The calibration weight therefore takes a quadratic profile, with polynomial order 2. Graph for $0 \le t \le T$ where $T = \frac{2u_y}{g} \approx 2.361$ s (total flight time):




**(d) Horizontal distance at maximum height**
At the highest point, the vertical velocity becomes zero. Using the SUVAT equation $v = u + at$ for vertical motion ($v_y = u_y - gt$):

$$0 = u_y - gt_{\text{top}} \implies t_{\text{top}} = \frac{u_y}{g}.$$

Substitute the values:

$$t_{\text{top}} = \frac{11.57}{9.8} \approx 1.181 \text{ s.}$$

**Horizontal distance:** Horizontal motion is uniform (no horizontal acceleration):

$$x(t) = u_x t.$$

Thus at maximum height:

$$x_{\text{top}} = u_x \times t_{\text{top}} \approx 13.788 \times 1.181 \approx 16.28 \text{ m.}$$

$$\boxed{x_{\text{top}} \approx 16.3 \text{ m}}.$$

**Note:** The horizontal motion reduces to $s = ut$ because the acceleration term $\frac{1}{2}at^2$ is zero (as stated in part (b), no horizontal force, hence $a_x = 0$).

**(e) Total horizontal range**
Total flight time: $T = \dfrac{2u_y}{g} \approx 2.361$ s. Range $R = u_x T \approx 13.788 \times 2.361 \approx 32.56$ m.
(Note the time of flight is simply twice the time to maximum height found in part (d), due to the symmetry of the parabolic trajectory.)

Alternatively, using the standard range formula:

$$R = \frac{u^2 \sin(2\theta)}{g} = \frac{18^2 \sin 80^\circ}{9.8} = \frac{324 \times 0.9848}{9.8} \approx 32.56 \text{ m.}$$

$$\boxed{R \approx 32.6 \text{ m}}.$$

**(f) Time spent above 2 m**

Solve $y(t) \geq 2$:

$$11.57t - 4.9t^2 \geq 2 \quad \Longrightarrow \quad 4.9t^2 - 11.57t + 2 \leq 0.$$

Quadratic roots:

$$t_1 \approx 0.1878 \text{ s}, \qquad t_2 \approx 2.1735 \text{ s}.$$

The weight is above 2 m between these two times. Duration above 2 m:

$$\Delta t = t_2 - t_1 \approx 2.1735 - 0.1878 = 1.9857 \text{ s}.$$

$$\boxed{\Delta t \approx 1.99 \text{ s}}.$$

The dashed horizontal line in the graph (c) at $y = 2$m crosses the parabola at these two instants; the projectile clears the 2m barrier for nearly 2 seconds.

8. An electric vehicle uses a supercapacitor bank of capacitance $C = 100\,\text{F}$ to store energy from regenerative braking. The bank charges through an internal resistance $R = 0.2\,\Omega$ with voltage

$$V(t) = V_0\left(1 - e^{-t/(RC)}\right), \quad V_0 = 50\,\text{V}.$$

(a) Identify the constant $a$ in $V = V_0(1 - e^{-at})$ and calculate its value with units.
(b) Find the charging current $I(t) = C\frac{dV}{dt}$ and evaluate the initial current.
(c) Determine the time constant $\tau = RC$. When does the current drop to half its initial value?
(d) Find $\frac{dI}{dt}$ and evaluate it at $t = \tau$.

## Solution:

**(a) Identify the constant $a$ and its value**

The voltage is given as $V = V_0(1 - e^{-at})$. Comparing with the expression $V(t) = V_0\left(1 - e^{-t/(RC)}\right)$ gives

$$a = \frac{1}{RC}.$$

Substitute $R = 0.2\ \Omega$, $C = 100$ F:

$$a = \frac{1}{0.2 \times 100} = 0.05 \text{ s}^{-1}.$$

$$\boxed{a = 0.05 \text{ s}^{-1}}.$$

**(b) Charging current $I(t)$ and initial current**

The current is given by $I(t) = C\,\dfrac{dV}{dt}$. First find $\dfrac{dV}{dt}$:

$$\frac{dV}{dt} = V_0 \cdot \frac{1}{RC}\,e^{-t/(RC)} = \frac{V_0}{RC}\,e^{-t/(RC)}.$$

Hence

$$I(t) = C \cdot \frac{V_0}{RC}\,e^{-t/(RC)} = \frac{V_0}{R}\,e^{-t/(RC)}.$$

With $V_0 = 50$ V and $R = 0.2\ \Omega$:

$$I(t) = \frac{50}{0.2}\,e^{-t/20} = 250\,e^{-t/20} \text{ A}.$$

The initial current at $t = 0$ is

$$I(0) = 250 \text{ A}.$$

$$\boxed{I(t) = 250\,e^{-t/20} \text{ A}, \qquad I(0) = 250 \text{ A}}.$$

**(c) Time constant and time to half current**

The time constant is

$$\tau = RC = 0.2 \times 100 = 20 \text{ s}.$$

$$\boxed{\tau = 20\text{ s}}.$$

The current drops to half its initial value when $I(t) = \dfrac{I(0)}{2} = 125$ A:

$$250\,e^{-t/20} = 125 \quad \Longrightarrow \quad e^{-t/20} = \frac{1}{2}.$$

Take natural logarithms:

$$-\frac{t}{20} = \ln\left(\frac{1}{2}\right) = -\ln 2 \quad \Longrightarrow \quad t = 20\ln 2\text{ s}.$$

$$\boxed{t = 20\ln 2\text{ s} \approx 13.86\text{ s}}.$$

**(d) Derivative of current and its value at $t = \tau$**

Differentiate $I(t) = \dfrac{V_0}{R}\,e^{-t/(RC)}$ with respect to $t$:

$$\frac{dI}{dt} = \frac{V_0}{R}\cdot\left(-\frac{1}{RC}\right)e^{-t/(RC)} = -\frac{V_0}{R^2C}\,e^{-t/(RC)}.$$

Alternatively, $\dfrac{dI}{dt} = -\dfrac{I(t)}{RC}$.
At $t = \tau = RC = 20$ s:

$$I(\tau) = \frac{V_0}{R}\,e^{-1} = 250\,e^{-1}\text{ A},$$

so

$$\left.\frac{dI}{dt}\right|_{t=\tau} = -\frac{I(\tau)}{RC} = -\frac{250\,e^{-1}}{20} = -12.5\,e^{-1}\text{ A/s}.$$

Numerically, $e^{-1} \approx 0.3679$, thus

$$\left.\frac{dI}{dt}\right|_{t=\tau} \approx -12.5 \times 0.3679 = -4.599\text{ A/s}.$$

$$\boxed{\left.\frac{dI}{dt}\right|_{t=\tau} = -\frac{V_0}{R^2C}\,e^{-1} = -12.5\,e^{-1}\text{ A/s} \approx -4.60\text{ A/s}}.$$

9. A nuclear fuel rod of length 0.2 m is cooled at both ends. Due to internal heat generation, the temperature is highest at the rod's centre. By symmetry, the temperature from one end to the centre can be modelled as:

$$T(x) = 300 + 10000x - 50000x^2, \quad 0 \le x \le 0.1$$

where $T$ is in °C and $x$ is the distance from one end in metres.

(a) Find $\frac{dT}{dx}$. What does this tell you about the temperature change as you move from the end towards the centre?

(b) Show that $\frac{dT}{dx} = 0$ at $x = 0.1$ m. Verify using the second derivative that this gives a maximum temperature and find the maximum temperature.

(c) The heat flux (in $\text{Wm}^{-2}$) is given by Fourier's law: $q = -k\frac{dT}{dx}$, with constant thermal conductivity $k = 3\,\text{Wm}^{-1}{}^\circ\text{C}^{-1}$. Calculate the heat flux at the end, and at the centre. Explain the physical meaning of the signs of these fluxes.

(d) In reality, the thermal conductivity of the fuel material is a function of temperature. Suppose that $k = 2 + 0.005T$ in $\text{Wm}^{-1}{}^\circ\text{C}^{-1}$. Find $\frac{dq}{dx}$ at $x = 0.05$ m.

## Solution:

### (a) First derivative and interpretation

$$\frac{dT}{dx} = 10000 - 100000x.$$

At the end $x = 0$: $\frac{dT}{dx} = 10000 > 0$. As we move from the end toward the centre, the derivative is positive but decreases linearly. This means the temperature **increases** as we move away from the end, but the rate of increase slows down. The temperature gradient becomes zero at the centre.

### (b) Maximum temperature at the centre

At $x = 0.1$:

$$\frac{dT}{dx} = 10000 - 100000(0.1) = 10000 - 10000 = 0.$$

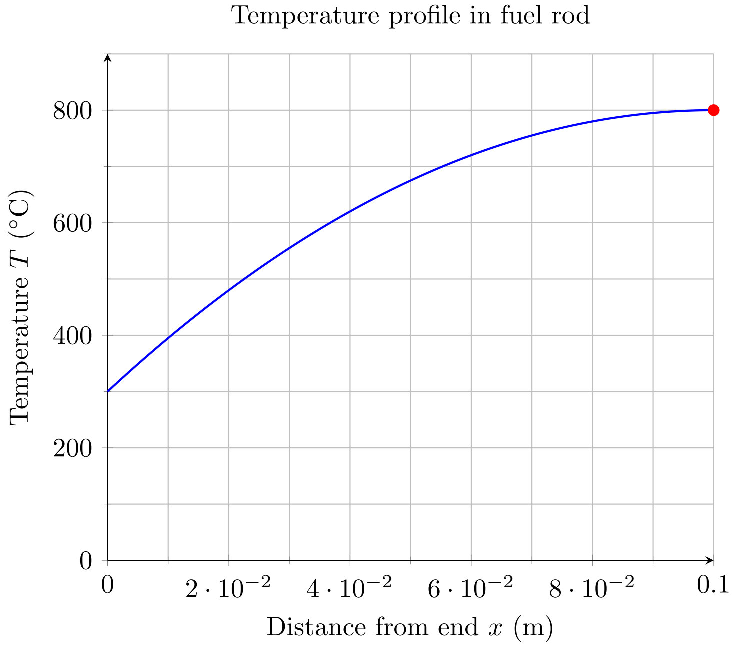
Hence there is a stationary point at the centre.
Second derivative:

$$\frac{d^2T}{dx^2} = -100000 < 0,$$

which confirms that the stationary point is a **maximum**.
Maximum temperature:

$$T_{\max} = T(0.1) = 300 + 10000(0.1) - 50000(0.1)^2 = 800\ ^\circ\text{C}.$$

$$\boxed{T_{\max} = 800\ ^\circ\text{C at } x = 0.1\text{ m}}.$$




### (c) Heat flux at the end and centre

Fourier's law: $q = -k\frac{dT}{dx}$, with constant $k = 3\ \text{Wm}^{-1}{}^\circ\text{C}^{-1}$.
At the end $x = 0$: $\frac{dT}{dx} = 10000\ ^\circ\text{C/m}$.

$$q(0) = -3 \times 10000 = -30000\ \text{W/m}^2.$$

At the centre $x = 0.1$: $\frac{dT}{dx} = 0$.

$$q(0.1) = -3 \times 0 = 0\ \text{W/m}^2.$$

$$\boxed{q(0) = -30000 \text{ W/m}^2, \qquad q(0.1) = 0 \text{ W/m}^2}.$$

**Physical meaning of the signs:** The negative sign at the end indicates that heat flows **opposite** to the direction of increasing $x$. Since $x$ increases from the end toward the centre, a negative heat flux means heat is flowing from the hotter interior **toward the cooler end** (i.e. in the negative $x$ direction). At the centre, the temperature gradient is zero, so there is no net heat flux – this is the symmetry point where heat flows out equally to both ends.

**(d) Heat flux derivative with temperature-dependent conductivity**

Now $k = 2 + 0.005T$ (units: $\text{Wm}^{-1}{}^\circ\text{C}^{-1}$). Still $q = -k\,\dfrac{dT}{dx}$. By applying the product rule, differentiate with respect to $x$:

$$\frac{dq}{dx} = -\frac{dk}{dx}\frac{dT}{dx} - k\frac{d^2T}{dx^2},$$

where $\dfrac{dk}{dx} = 0.005\,\dfrac{dT}{dx}$.

At $x = 0.05$ m:

$$T = 300 + 10000(0.05) - 50000(0.05)^2 = 675\ ^\circ\text{C},$$

$$\frac{dT}{dx} = 10000 - 100000(0.05) = 5000\ ^\circ\text{C/m},$$

$$\frac{d^2T}{dx^2} = -100000\ ^\circ\text{C/m}^2,$$

$$k = 2 + 0.005(675) = 5.375 \text{ Wm}^{-1}{}^\circ\text{C}^{-1},$$

$$\frac{dk}{dx} = 0.005 \times 5000 = 25 \text{ Wm}^{-1}{}^\circ\text{C}^{-1}/\text{m} \ \ (\text{or } 25 \text{ Wm}^{-2}{}^\circ\text{C}^{-1}).$$

Now substitute into $\dfrac{dq}{dx}$:

$$\frac{dq}{dx} = -(25)(5000) - (5.375)(-100000) = 412500 \text{ W/m}^3.$$

$$\boxed{\left.\frac{dq}{dx}\right|_{x=0.05} = 412500 \text{ W/m}^3}.$$

This positive value indicates that at $x = 0.05$ m, the heat flux is **increasing** with distance from the end; physically, the rate of heat transfer per unit area grows as we move toward the centre, reflecting the internal heat generation within the rod.

10. In a simplified model, the circumferential stress $\sigma$ (in MPa) at radius $r$, in metres, in a rotating disc is given by:

$$\sigma(r) = \frac{300r^2}{1+4r^2}, \quad 0 \le r \le 1.$$

(a) Use graphing software to plot $\sigma$ against $r$ on $0 \le r \le 1$.

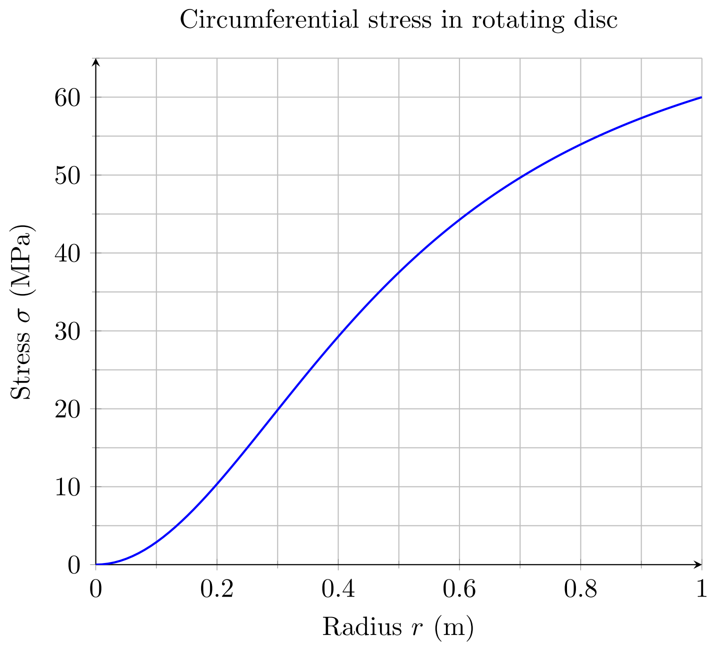
(b) Differentiate to find $\frac{d\sigma}{dr}$ and simplify as far as possible. State the units.

(c) Find the radius at which $\sigma$ is increasing most rapidly with $r$.

(d) Engineers are most concerned about regions where a small increase in radius causes a large increase in stress. Using your derivative, identify an interval of $r$ where $\left|\frac{d\sigma}{dr}\right|$ is "large" and briefly explain why this might influence design or material choice.

## Solution:

**(a) Plot of $\sigma$ against $r$**




**(b) Derivative $\frac{d\sigma}{dr}$ and simplification**

Using the quotient rule:

$$\frac{d\sigma}{dr} = \frac{(600r)(1+4r^2) - 300r^2(8r)}{(1+4r^2)^2} = \frac{600r + 2400r^3 - 2400r^3}{(1+4r^2)^2} = \frac{600r}{(1+4r^2)^2}.$$

$$\boxed{\frac{d\sigma}{dr} = \frac{600r}{(1+4r^2)^2}}.$$

**Units:** $\sigma$ is in MPa, $r$ in m, so $\frac{d\sigma}{dr}$ is in MPa/m.

**(c) Radius at which stress increases most rapidly**

We need the maximum of $\frac{d\sigma}{dr} = f(r) = \frac{600r}{(1+4r^2)^2}$ on $0 \leq r \leq 1$. Differentiate $f(r)$ using the quotient rule (or rewrite as $600r(1+4r^2)^{-2}$ and use product/chain rule).

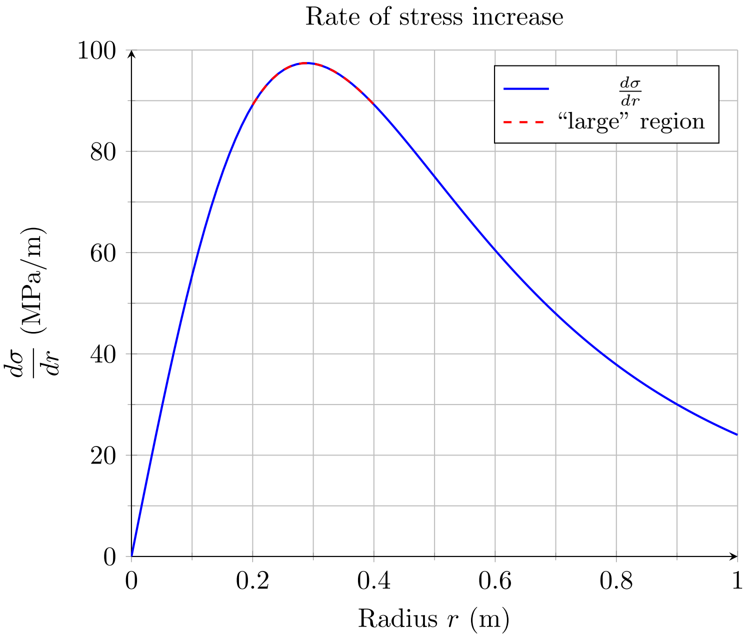
$$f'(r) = 600\,\frac{(1+4r^2)^2 - r\cdot 2(1+4r^2)\cdot 8r}{(1+4r^2)^4} = 600\,\frac{(1+4r^2)\big[(1+4r^2) - 16r^2\big]}{(1+4r^2)^4} = \frac{600\,(1-12r^2)}{(1+4r^2)^3}.$$

Set $f'(r) = 0$: $1 - 12r^2 = 0 \;\Rightarrow\; r = \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}} \approx 0.2887$ m. Since $f'(r) > 0$ for $r < \frac{1}{2\sqrt{3}}$ and $f'(r) < 0$ for $r > \frac{1}{2\sqrt{3}}$, this critical point is a maximum. Thus stress increases most rapidly at

$$\boxed{r = \frac{1}{2\sqrt{3}}\text{ m} \approx 0.289\text{ m}}.$$

**(d) Region of large $\left|\frac{d\sigma}{dr}\right|$ and design implication**

The derivative $\frac{d\sigma}{dr} = \frac{600r}{(1+4r^2)^2}$ is positive throughout $[0,1]$, so $\left|\frac{d\sigma}{dr}\right| = \frac{d\sigma}{dr}$. Its maximum occurs at $r \approx 0.289$ m, and it decreases as $r$ moves away from this value. A reasonable interval where $\left|\frac{d\sigma}{dr}\right|$ is "large" could be, for example, $0.2 \leq r \leq 0.4$ (centred on the maximum). Within this range, the derivative remains close to its peak value.




**Why this matters for design/material choice:** In the region where $\left|\frac{d\sigma}{dr}\right|$ is large, a small uncertainty or manufacturing tolerance in the radius will cause a disproportionately large change in stress. This means that if the disc is to operate in that radial zone, engineers must:

- Specify tighter manufacturing tolerances to avoid unexpected stress concentrations,
- Choose materials with sufficient strength and safety margins to accommodate possible stress variations,
- Possibly redesign the disc profile (e.g., varying thickness) to reduce the stress gradient in that critical region.