Differentiation – Prereading

1. The Idea of a Small Change

Imagine you are watching a car speedometer. The needle shows the speed at an instant – that is an **instantaneous rate**. But how do we measure it? We could time how far the car goes in one second, but that's an average over a second. To get the exact speed at a precise moment, we need to consider an **infinitesimally small** time interval.

In mathematics, we denote a tiny change in a quantity x by dx (read "dee x"). Think of it as a "small amount of x". For example, one minute is a small amount of an hour; one second is a small amount of a minute. We can talk about **orders of smallness** – a second is a second-order small fraction of an hour because it is $\frac{1}{3600}$ of an hour, whereas a minute is $\frac{1}{60}$ (first order). When we take ratios of small changes, the higher-order terms become negligible. This idea leads us to the derivative.

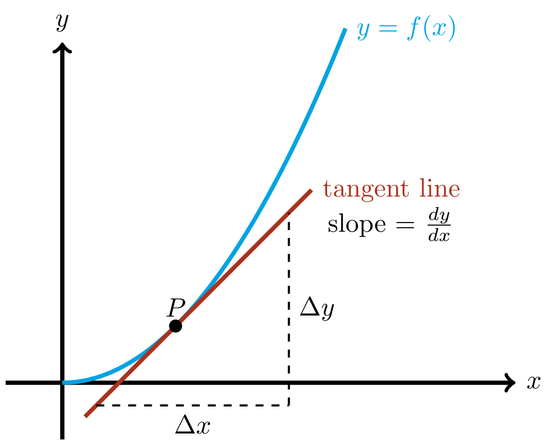
> **Historical Note:** In Latin, the phrase *pars minuta prima* means "the first small part", and a further division is called *pars minuta secunda*, meaning "the second small part". Correspondingly, in English, **minutes** and **seconds** were originally referred to as "minutes" and "second minutes."

2. The Derivative as a Rate of Change

If a quantity y depends on x (written $y = f(x)$), then the **derivative** $\frac{dy}{dx}$ measures how fast y changes as x changes. Formally,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x},$$

where Δy is the change in y corresponding to a small change Δx in x. In words, it is the **instantaneous rate of change** of y with respect to x. This can also be written as $f'(x)$ (read "f prime of x").
Graphically, $\frac{dy}{dx}$ (or $f'(x)$) is the slope of the tangent line to the curve $y = f(x)$ at a given point, P.



3. Basic Differentiation Rules

Instead of using limits every time, we use standard rules.

Rule Name	General Rule	Example
Power Rule	$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(x^5) = 5x^4$
Constant Multiple Rule	$\frac{d}{dx}\big(c\,f(x)\big) = c\,f'(x)$	$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$
Sum/Difference Rule	$\frac{d}{dx}\big(f(x) \pm g(x)\big) = f'(x) \pm g'(x)$	$\frac{d}{dx}(x^2 + 3x) = 2x + 3$
Derivative of a Constant	$\frac{d}{dx}(c) = 0$	$\frac{d}{dx}(5) = 0$

4. Using Derivatives to Find Maxima and Minima

Many engineering problems ask for the largest or smallest value of a function — for example, to maximise an output parameter or minimise the cost of materials. At a maximum or minimum point, the tangent to the curve is horizontal. A horizontal line has equation $y = c$ (constant), and its slope is zero. Therefore, at any maximum or minimum, the derivative must be zero: $f'(x) = 0$.
The key steps to find maxima and minima are:

1. Find the derivative $f'(x)$.
2. Solve $f'(x) = 0$ to locate *stationary points* (where the tangent is horizontal).
3. Classify each stationary point as a maximum, minimum, or neither. You can use the **second derivative** (see below) or alternatively, examine the sign of $f'(x)$ just before and after the point.
4. If the function is defined on a **closed interval** (e.g., $a \le x \le b$), also evaluate f at the endpoints $x = a$ and $x = b$. These endpoints could potentially give a maximum or minimum value, even if the slope is not zero there.

5. Second Derivatives and Concavity

The second derivative is simply the derivative of the first derivative. It can be written as $f''(x)$ or $\frac{d^2y}{dx^2}$. While the first derivative tells us about the slope (whether the function is increasing or decreasing), the second derivative tells us about the **curvature** or **concavity** of the graph.

- If $f''(x) > 0$, the graph is concave up (like a cup) and stationary points are minima.
- If $f''(x) < 0$, the graph is concave down (like a cap) and stationary points are maxima.
- If $f''(x) = 0$, the test is inconclusive – you may need another method.

6. Kinematics: Position, Velocity, Acceleration

When a quantity varies with time, derivatives give the rates of change. In linear motion:

$$\text{Velocity: } v(t) = \frac{ds}{dt}, \qquad \text{Acceleration: } a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

The same ideas apply to rotational motion: if $\theta(t)$ is the angular position (in radians), then

$$\text{Angular velocity: } \omega(t) = \frac{d\theta}{dt}, \qquad \text{Angular acceleration: } \alpha(t) = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

7. Differentiating Trigonometric and Exponential Functions

Beyond polynomials, many engineering models involve sine, cosine, and exponential functions. Here are the basic derivatives:

Function	Derivative	Example
$\sin x$	$\cos x$	$\frac{d}{dx}(5\sin x) = 5\cos x$
$\cos x$	$-\sin x$	$\frac{d}{dx}(3\cos 2x) = -6\sin 2x$ (using chain rule)
e^x	e^x	$\frac{d}{dx}(e^t) = e^t$
e^{kx}	ke^{kx}	$\frac{d}{dx}(2e^{-3t}) = -6e^{-3t}$
$\ln x$	$\frac{1}{x}$	$\frac{d}{dx}(4\ln x) = \frac{4}{x}$

8. Advanced Differentiation Rules

For combinations of functions, we need three powerful rules.

8.1 Product Rule

If $f(x) = u(x)\,v(x)$, then

$$f'(x) = u'(x)v(x) + u(x)v'(x).$$

Example: Differentiate $y = x^2 \sin x$.

$$\frac{dy}{dx} = (2x)(\sin x) + (x^2)(\cos x) = 2x\sin x + x^2\cos x.$$

8.2 Quotient Rule

If $f(x) = \dfrac{u(x)}{v(x)}$ (with $v(x) \neq 0$), then

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}.$$

Example: Differentiate $y = \dfrac{x}{1+x^2}$.

$$\frac{dy}{dx} = \frac{(1)(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}.$$

8.3 Chain Rule

The chain rule handles compositions: if $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

Think: "differentiate the outer function, leaving the inner alone, then multiply by the derivative of the inner."

Example: Differentiate $y = e^{-t/5}$.

$$\frac{dy}{dt} = e^{-t/5} \cdot \left(-\frac{1}{5}\right) = -\frac{1}{5}e^{-t/5}.$$

(Notice that this is simply an application of the standard rule $\frac{d}{dx}e^{kx} = ke^{kx}$ from Section 7, with $k = -\frac{1}{5}$. The chain rule provides the justification for this rule.)

These three rules, together with the basic derivatives, allow us to differentiate almost any function encountered in engineering.