INSTITUTE OF MATHEMATICS & ITS APPLICATIONS

Differentiation – Problems

1. In an electrical circuit test for a power supply, the voltage drop V (in volts) across a component varies with current I (in amps) is $V = 10I + 0.5I^2$. For this non-linear component, the instantaneous resistance (R) is defined as the slope of the $V - I$ curve. Find R when the current through the component is 0.1 amps.

2. A chemical engineer is designing a rectangular reaction tank next to a factory wall, with one long side running along the wall (no lining required there). The total length of lining material available for the remaining three sides is 16m. Let the width perpendicular to the wall be x m.
   (a) Express the tank's base area A in terms of x only.
   (b) Differentiate A with respect to x and find the dimensions that maximise the area.
   (c) By considering the second derivative or otherwise, verify that your dimensions give a maximum.
   (d) Sketch the graph of A against x. State the range of values for x and verify that your maximum lies within it.

3. A rigid arm holds a set of traffic lights out over a junction. Engineers model the vertical deflection y (in mm) of the tip of the arm under load by:

$$y(F) = 0.02F^2 + 0.3F, \quad F \geq 0$$

   where F is the total equipment load at the tip in kN (lights, cameras, signage, etc.).

   (a) Plot a graph of y against F for $0 \leq F \leq 20$.
   (b) Find $\frac{dy}{dF}$, state the units and interpret this derivative physically.
   (c) Find $\frac{d^2y}{dF^2}$, state the units and explain what the sign implies about how the arm's stiffness changes as the load due to the lights increases.
   (d) At what value of F does an additional 1 kN of load cause the greatest extra deflection on the rigid arm?

4. In a flywheel energy storage system, a rotating flywheel stores kinetic energy. The angular position (in radians) of the flywheel during a testing cycle is given by:

$$\theta(t) = 0.1t^4 - 2t^3 + 10.5t^2, \quad 0 \leq t \leq 10$$

   where t is time in seconds.
   (a) Find the angular velocity $\omega(t)$ and angular acceleration $\alpha(t)$. State the units.
   (b) Determine all times t in $[0, 10]$ when the flywheel is momentarily at rest.
   (c) For each value of time found in (b), decide whether the flywheel reverses direction or momentarily stops and then continues in the same direction. Explain your reasoning.
   (d) Find the time(s) when the angular acceleration is zero. What does this imply about the rotational motion?
   (e) Use a sketch or a graphing software plot to illustrate the relationship between the angular displacement, velocity, and acceleration curves. Relate this to your answer in part (d).

5. Water is draining from a conical tank with vertex angle 60°. At any time t seconds, let the radius of the surface be r metres and the height of water be h metres. The volume of water in the tank is:

$$V = \frac{1}{3}\pi r^2 h$$

   and for this tank, $r = h\tan(30°) = \frac{h}{\sqrt{3}}$.

(a) Show that $V = \frac{\pi}{9}h^3$.
(b) Water drains at a constant rate of 0.02 $\mathrm{m^3s^{-1}}$. Find the surface drop rate, $\frac{dh}{dt}$, when $h = 1.5\,\mathrm{m}$ and give its units.
(c) Explain the physical meaning of your result from (b) in relation to its sign and magnitude.
(d) Find the depth h_d where the magnitude of the surface drop rate $\left|\frac{dh}{dt}\right|$ is twice the value found in (b).
(e) The tank starts full at $h = 3m$. As h approaches zero, explain whether $\left|\frac{dh}{dt}\right|$ increases or decreases. Interpret what your answer means for the engineer designing this drainage system.

6. During the testing of an electromagnetic field sensor, the measured electric and magnetic fields are found to be:
$$E(t) = \sin t, \qquad B(t) = \cos t.$$
Maxwell's equations in free space relate these fields by:
$$E = -\frac{dB}{dt} \quad \text{and} \quad B = \frac{dE}{dt}.$$
Verify that the measured fields satisfy both Maxwell's equations.

7. A quality control technician launches a small calibration weight from a testing rig to verify it clears a safety barrier during automated assembly. The weight is projected from ground level with an initial speed of $18\,\mathrm{ms^{-1}}$ at 40° above the horizontal. Assume that the acceleration due to gravity $g = 9.8\,\mathrm{ms^{-2}}$.
(a) State the horizontal and vertical components of the initial velocity.
(b) Assuming no air resistance, list all forces acting on the calibration weight and their respective directions.
(c) Express the vertical height y metres at time t seconds. Use a graphing software to show the relationship between y and t and state the polynomial order of this equation.
(d) Find the horizontal distance from launch at maximum height.
(e) Find the total horizontal range before it returns to ground.
(f) For installation planning, engineers need the calibration weight to remain at least 2 m above ground while passing over the safety barrier. Using your model $y(t)$, find how long the component stays above 2 m.

8. An electric vehicle uses a supercapacitor bank of capacitance $C = 100\,\mathrm{F}$ to store energy from regenerative braking. The bank charges through an internal resistance $R = 0.2\,\Omega$ with voltage
$$V(t) = V_0\left(1 - e^{-t/(RC)}\right), \quad V_0 = 50\,\mathrm{V}.$$
(a) Identify the constant a in $V = V_0(1 - e^{-at})$ and calculate its value with units.
(b) Find the charging current $I(t) = C\frac{dV}{dt}$ and evaluate the initial current.
(c) Determine the time constant $\tau = RC$. When does the current drop to half its initial value?
(d) Find $\frac{dI}{dt}$ and evaluate it at $t = \tau$.

9. A nuclear fuel rod of length 0.2 m is cooled at both ends. Due to internal heat generation, the temperature is highest at the rod's centre. By symmetry, the temperature from one end to the centre can be modelled as:
$$T(x) = 300 + 10000x - 50000x^2, \quad 0 \le x \le 0.1$$
where T is in °C and x is the distance from one end in metres.
(a) Find $\frac{dT}{dx}$. What does this tell you about the temperature change as you move from the end towards the centre?
(b) Show that $\frac{dT}{dx} = 0$ at $x = 0.1$ m. Verify using the second derivative that this gives a maximum temperature and find the maximum temperature.
(c) The heat flux (in $\mathrm{Wm^{-2}}$) is given by Fourier's law: $q = -k\frac{dT}{dx}$, with constant thermal conductivity $k = 3\,\mathrm{Wm^{-1}{}^\circ C^{-1}}$. Calculate the heat flux at the end, and at the centre. Explain the physical meaning of the signs of these fluxes.
(d) In reality, the thermal conductivity of the fuel material is a function of temperature. Suppose that $k = 2 + 0.005T$ in $\mathrm{Wm^{-1}{}^\circ C^{-1}}$. Find $\frac{dq}{dx}$ at $x = 0.05$ m.

10. In a simplified model, the circumferential stress σ (in MPa) at radius r, in metres, in a rotating disc is given by:

$$\sigma(r) = \frac{300r^2}{1+4r^2}, \quad 0 \le r \le 1.$$

(a) Use graphing software to plot σ against r on $0 \le r \le 1$.
(b) Differentiate to find $\frac{d\sigma}{dr}$ and simplify as far as possible. State the units.
(c) Find the radius at which σ is increasing most rapidly with r.
(d) Engineers are most concerned about regions where a small increase in radius causes a large increase in stress. Using your derivative, identify an interval of r where $\left|\frac{d\sigma}{dr}\right|$ is "large" and briefly explain why this might influence design or material choice.