

EMS412U Statics Revision Workbook (Weeks 5-9)

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1. Forces and Moments

Forces and moments are fundamental concepts in engineering mechanics. They describe how loads are applied to structures and how bodies respond to these loads. Understanding these concepts is essential for analysing the stability and strength of engineering systems.

1.1 Forces

A force is a vector quantity that represents the interaction between two bodies. It has both magnitude and direction, and its effect depends on where and how it is applied.

Scalar and Vector Quantities

All physical quantities in mechanics are measured using either a scalar or a vector.

Scalar Quantity: A scalar is any positive or negative physical quantity that can be specified by its magnitude alone. Examples include mass, temperature, and speed.

Vector Quantity: A vector is any physical quantity that requires both a magnitude and a direction for its complete description. Examples include force, velocity, and acceleration.

Vector Representation of Forces

A force vector in 2D can be written as the sum of its component vectors:

$$\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y$$

Each component is a scalar multiplied by a unit vector:

$$\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j}$$

The relationship between the components and the magnitude is:

$$F_x = F \cos \theta, \quad F_y = F \sin \theta, \quad F = \sqrt{F_x^2 + F_y^2}, \quad \theta = \tan^{-1} \left(\frac{F_y}{F_x} \right)$$

Newton's Second Law

The force acting on a body is related to its mass and acceleration by Newton's Second Law:

$$\mathbf{F} = m\mathbf{a}$$

Resultant Forces

Resultant forces can be calculated using vector addition. All vectors obey the parallelogram law of addition.

To add two vectors **A** and **B** using the parallelogram method:

1. Join the tails of the vectors at a point to make them concurrent.
2. From the head of **B**, draw a line parallel to **A**.
3. From the head of **A**, draw a line parallel to **B**.
4. The diagonal of the parallelogram represents the resultant vector $\mathbf{R} = \mathbf{A} + \mathbf{B}$.

Alternatively, the triangle law can be used, though care must be taken to preserve the correct line of action.

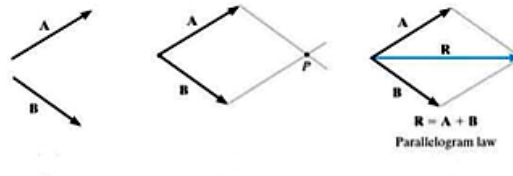


Figure 1.1: Parallelogram law of vector addition.

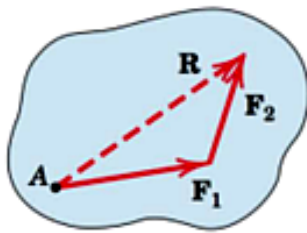


Figure 1.2: Triangle law (line of action may be lost).

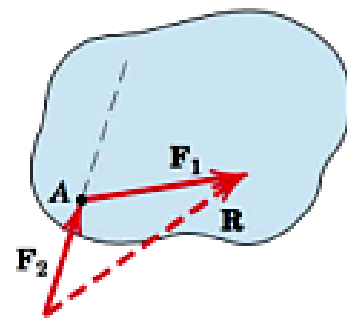
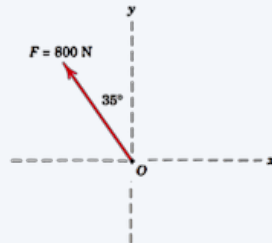


Figure 1.3: Correct resultant with preserved line of action.

Example 1.1 Expressing a Force in Vector Form Either the figure is wrong or the answer is wrong. The vector is not making a 35 degrees angle with the negative x -axis but the positive y -axis

Problem: A force $\mathbf{F}$ has a magnitude of 800 N. Express $\mathbf{F}$ as a vector in terms of unit vectors $\mathbf{i}$ and $\mathbf{j}$. Identify the x and y scalar components of $\mathbf{F}$.



Solution:

Step 1: Identify the angle from the given diagram.

The force makes an angle of 35° with the negative x -axis.

Therefore, the angle from the positive x -axis is $180^\circ - 35^\circ = 145^\circ$.

Step 2: Calculate the scalar components.

$$F_x = 800 \cos(145^\circ) = -459 \text{ N}$$

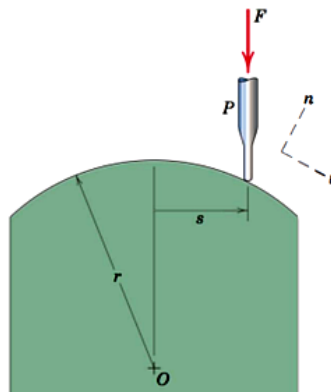
$$F_y = 800 \sin(145^\circ) = 655 \text{ N}$$

Step 3: Express the force in vector form.

$$\mathbf{F} = -459\mathbf{i} + 655\mathbf{j} \text{ N}$$

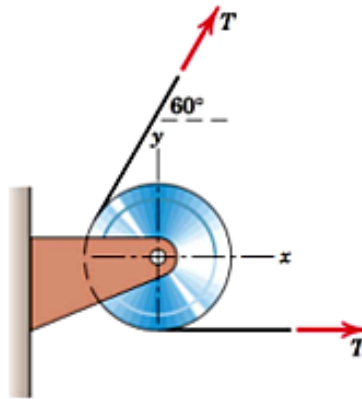
$$F_x = -459 \text{ N}, \quad F_y = 655 \text{ N}$$

Q1. A small probe P is gently forced against a circular surface with a vertical force $\mathbf{F}$ as shown. Determine the normal and tangential components of this force as functions of the horizontal position s .



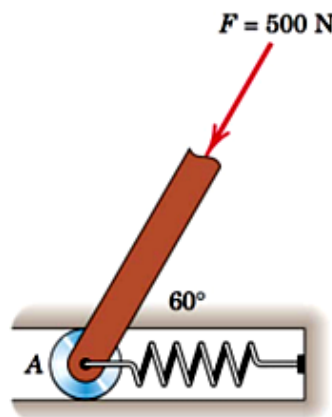
$$\left[\text{ans: } F_t = \frac{Fs}{r}, \quad F_n = \frac{F\sqrt{r^2 - s^2}}{r} \right]$$

Q2. If tensions T in a pulley cable are 400 N, express in vector notation the force $\mathbf{R}$ exerted on the pulley by the two tensions. Determine the magnitude of $\mathbf{R}$.



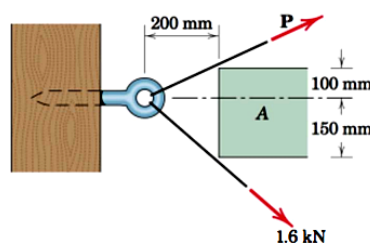
$$[\text{ans: } \mathbf{R} = 600\mathbf{i} + 346\mathbf{j} \text{ N}, \quad R = 693 \text{ N}]$$

Q3. Determine the magnitude F_s of the tensile spring force such that the resultant of F_s and $\mathbf{F}$ is a vertical force. Determine the magnitude R of this vertical resultant force.



$$[\text{ans: } F_s = 250 \text{ N}, \quad R = 433 \text{ N}]$$

Q4. A spike is to be removed from timber by applying a force along its horizontal axis. An obstruction prevents direct access, so two forces—one 1.6 kN and another P —are applied by cables. Find the magnitude of P necessary to ensure the resultant T is directed along the spike. Also find T .



[ans: $P = 2.15 \text{ kN}$, $T = 3.2 \text{ kN}$]

1.2 Moments

When a force is applied to a body, it can create a tendency for the body to rotate. This rotational tendency is called the **moment** of a force, or simply the **moment**.

Consider a wrench being used to unscrew a bolt. A force applied perpendicular to the handle creates a tendency to rotate the bolt around the vertical axis. The magnitude of this tendency depends on both:

- The magnitude of the force, F
- The perpendicular distance, d , from the line of action of the force to the point of rotation

The moment $\mathbf{M}$ about point O is defined as:

$$M = Fd$$

The units of moment are Newton-metres (N·m).

If the force is applied at an angle θ that is not perpendicular, the perpendicular distance becomes:

$$d' = d \sin \theta$$

Thus:

$$M = Fd \sin \theta$$

For a general force $\mathbf{F}$ at a perpendicular distance d from point O , the moment about O is $M = Fd$.

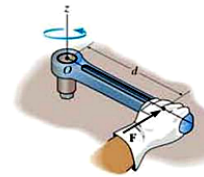
For 2D problems where all forces lie within the same plane, the resultant moment about a point is determined by finding the algebraic sum of all the moments.

The standard convention is:

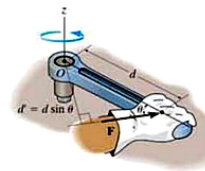
- Counterclockwise moments are positive
- Clockwise moments are negative

$$\Sigma M_O = F_1 d_1 - F_2 d_2 + F_3 d_3$$

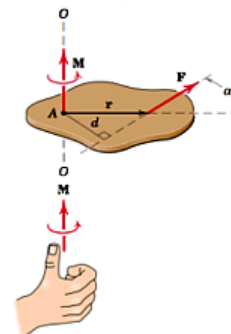
The diagram on the right illustrates how the algebraic sum is applied to a beam with multiple forces. Each force contributes a moment equal to its magnitude multiplied by its perpendicular distance from point O , with the sign determined by the direction of rotation.



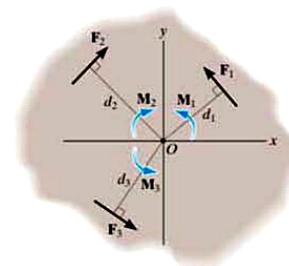
Force perpendicular to the handle.



Force at a non-perpendicular angle.



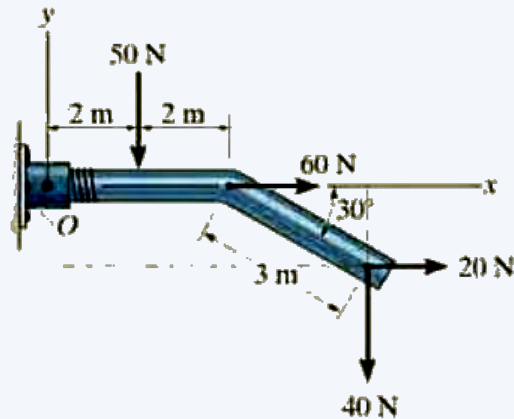
General definition of moment about point O .



Resultant moment as the algebraic sum of individual moments.

Example 1.2 Finding the Resultant Moment of Several Forces

Problem: Determine the resultant moment of the four forces acting on the rod shown.



Solution:

Step 1: Calculate the moment of each force about point O .

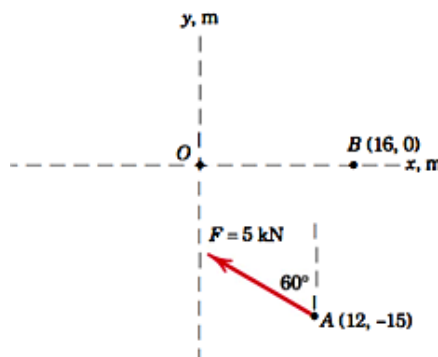
Force	Moment Arm	Moment
$F = 50 \text{ N}$	$d = 2 \text{ m}$ (clockwise)	$-100 \text{ N}\cdot\text{m}$
$F = 60 \text{ N}$	$d = 0 \text{ m}$ (at point O)	$0 \text{ N}\cdot\text{m}$
$F = 20 \text{ N}$	$d = 3 \sin 30^\circ \text{ m}$ (anticlockwise)	$+30 \text{ N}\cdot\text{m}$
$F = 40 \text{ N}$	$d = (4 + 3 \cos 30^\circ) \text{ m}$ (clockwise)	$-264 \text{ N}\cdot\text{m}$

Step 2: Sum all moments.

$$M_{R_O} = -100 + 0 + 30 - 264 = -334 \text{ N}\cdot\text{m}$$

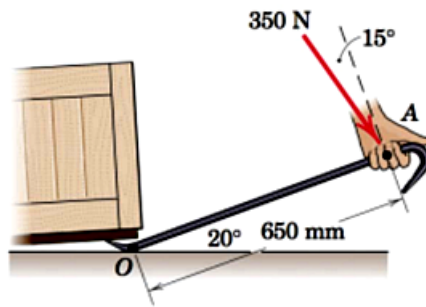
$$M_{R_O} = -334 \text{ N}\cdot\text{m} \text{ (clockwise)}$$

Q1. Determine the moments of the 5-kN force about point O and about point B .



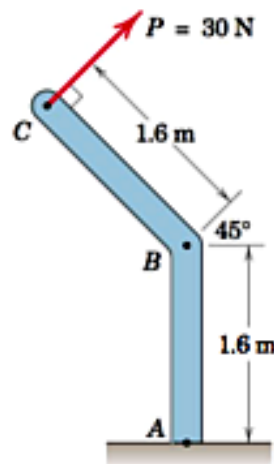
$$[\text{ans: } M_O = 35.0 \text{ kN}\cdot\text{m}, \quad M_B = 75.0 \text{ kN}\cdot\text{m}]$$

Q2. A worker uses a crowbar to lift the corner of a hot tub for maintenance. Determine the moment made by the 350 N force about point O . Neglect the small thickness of the crowbar.



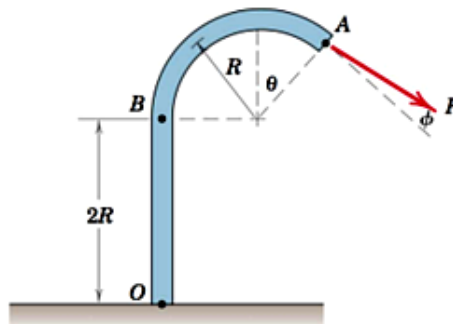
[ans: $M_O = 220 \text{ N}\cdot\text{m}$]

Q3. The 30 N force P is applied perpendicular to the portion BC of the bent bar. Determine the moment of P about point B and about point A .



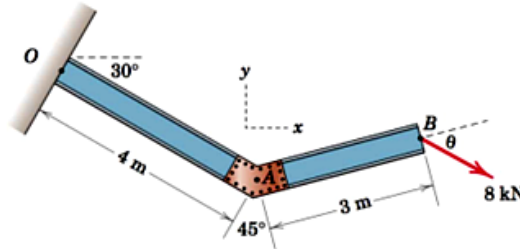
[ans: $M_B = 48 \text{ N}\cdot\text{m}$, $M_A = 81.9 \text{ N}\cdot\text{m}$]

Q4. Determine the general expressions for the moments of $\mathbf{F}$ about point B and about point O . Evaluate your expressions for $F = 750 \text{ N}$, $R = 2.4 \text{ m}$, $\theta = 30^\circ$, and $\phi = 15^\circ$.



[ans: $M_B = 2200 \text{ N}\cdot\text{m}$, $M_O = 5680 \text{ N}\cdot\text{m}$]

- Q5.** The bent cantilever beam is acted upon by an 8 kN force at B . If the angle $\theta = 35^\circ$, determine:
- The moment M_O of the force about point O
 - The moment M_A of the force about point A
- What value(s) of θ ($0^\circ < \theta < 360^\circ$) will result in the maximum possible moment about point O , and what is the magnitude of the moment at those orientations?



[ans: $M_O = -8.21 \text{ kN}\cdot\text{m}$, $M_A = -13.77 \text{ kN}\cdot\text{m}$, $\theta = 115.9^\circ, 296^\circ$, $(M_O)_{\max} = 51.8 \text{ kN}\cdot\text{m}$]

1.3 Free Body Diagrams and Supports

Rigid Bodies

A **rigid body** is a body that does not bend, stretch, or deform when loads are applied to it. While this is an idealisation, it allows us to analyse forces acting on a body by ignoring small deformations.

A body subjected to external forces and moments experiences effects from gravitational, electrical, magnetic, or contact forces from adjacent bodies. Internal forces between particles within the body act in equal and opposite collinear pairs and therefore cancel out—this is a consequence of Newton's Third Law.

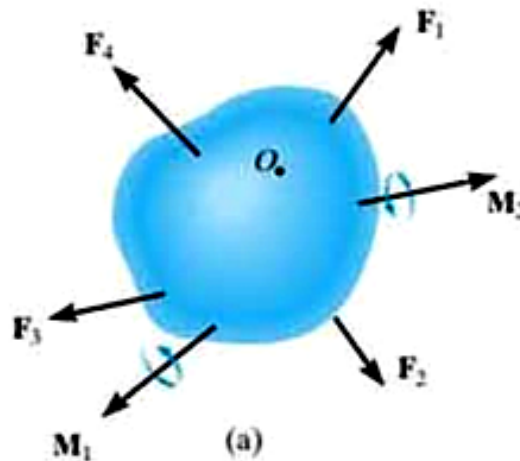


Figure 1.4: External and internal forces on a rigid body.

Free Body Diagrams

A **free body diagram** (FBD) is a sketch of the outlined shape of a body, represented as isolated or "free" from its surroundings. On this sketch, all external forces and couple moments acting on the body must be shown.

Steps for Drawing a Free Body Diagram

1. Determine the point or body of interest.
2. Cut the body away from the system and isolate it.
3. Draw the body as a single point or outline.
4. Add all forces acting on that body. Account for the type of forces—for example, ropes and chains **in taut** always have tension forces.
5. Label all components of the free body diagram.
6. Resolve all forces that are at an angle into Cartesian components. All forces must act away from the point.
7. Write the equations of equilibrium:

$$\Sigma F_x = 0, \quad \Sigma F_y = 0, \quad \Sigma M_z = 0$$

8. Solve for all unknowns algebraically.

Types of Supports

Different types of supports provide different reactions:

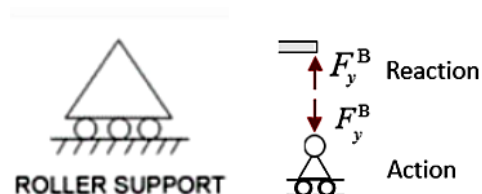
- If a support prevents translation in a direction, a force is developed in that direction.
- If rotation is prevented, a moment is exerted on the body.

Roller Support

is it possible to add the beam to the roller/pin support in the figure - it is currently not very clear

A roller support:

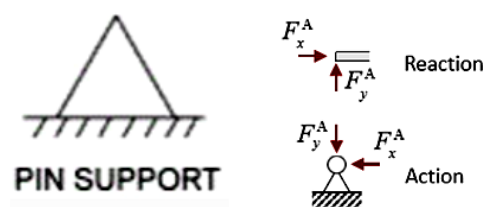
- Cannot move vertically
- Can move horizontally
- Can rotate
- 1 degree of freedom restricted = 1 reaction



Pin Support

A pin support:

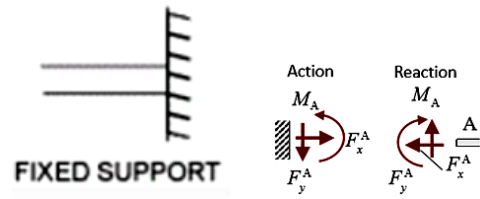
- Cannot move vertically
- Cannot move horizontally
- Can rotate
- 2 degrees of freedom restricted = 2 reactions



Fixed Support

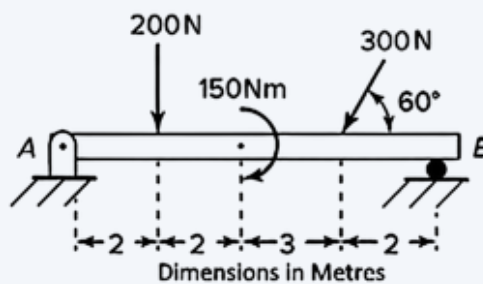
A fixed support:

- Cannot move vertically
- Cannot move horizontally
- Cannot rotate
- 3 degrees of freedom restricted = 3 reactions



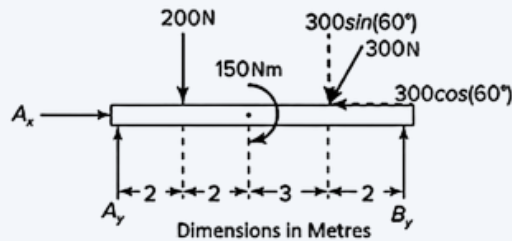
Example 1.3 Finding Support Reactions

Problem: Find the reactions at the supports for the simply supported beam shown.



Solution:

Step 1: Draw the free body diagram.



The beam is supported by a pin at A and a roller at B. The pin provides reactions A_x and A_y , and the roller provides a reaction B_y .

Step 2: Write the equations of equilibrium. Take moments about point A (where the most unknowns act):

$$\Sigma M_A = 0 = -200(2) - 150 - 300 \sin 60^\circ (7) + B_y(9)$$

$$\Sigma F_x = 0 = A_x - 300 \cos 60^\circ$$

$$\Sigma F_y = 0 = -200 - 300 \sin 60^\circ + A_y + B_y$$

Step 3: Solve for the unknowns.

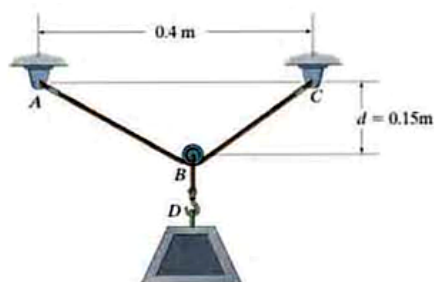
$$B_y = 263.18 \text{ N}$$

$$A_x = 150 \text{ N}$$

$$A_y = 196.62 \text{ N}$$

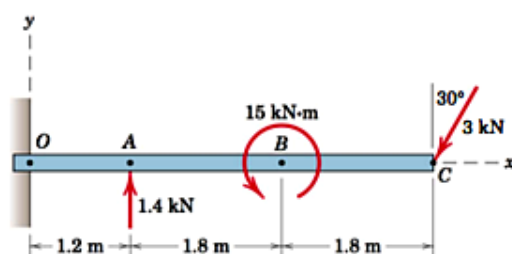
$$A_x = 150 \text{ N}, \quad A_y = 196.62 \text{ N}, \quad B_y = 263.18 \text{ N}$$

Q1. If the 5 kg block is suspended from pulley B and the sag of the cord is $d = 0.15 \text{ m}$, determine the force in cord ABC . Neglect the size of the pulley.



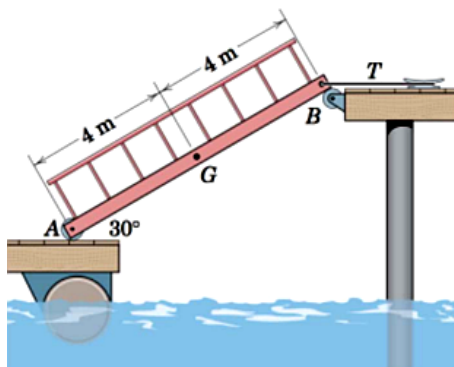
$$[\text{ans: } T = 40.9 \text{ N}]$$

Q2. The 500 kg uniform beam is subjected to the three external loads shown. Compute the reactions at the support point O .



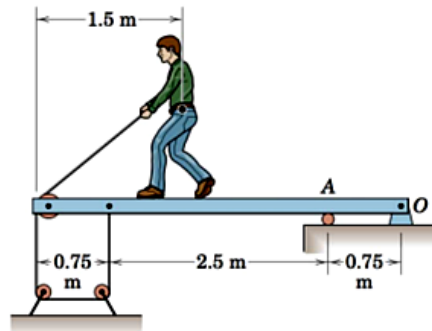
$$[\text{ans: } O_x = 1500 \text{ N}, \quad O_y = 6100 \text{ N}, \quad M_O = 7560 \text{ N}\cdot\text{m}]$$

Q3. A pier to a float is supported by two rollers. If the centre of mass of the 300 kg walkway is at G , calculate the tension T in the horizontal cable and find the force under the roller at A .



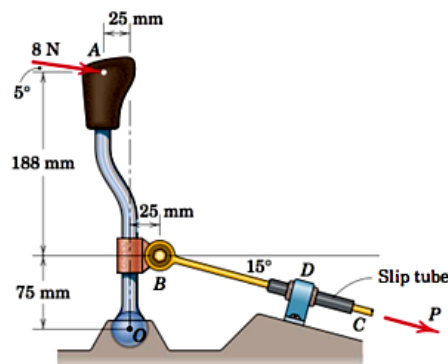
$$[\text{ans: } T = 850 \text{ N}, \quad A = 1472 \text{ N}]$$

Q4. To test the deflection of the uniform 100 kg beam, the 50 kg boy exerts a pull of 150 N on the rope rigged as shown. Work out the force supported by the pin at hinge O .



[ans: $O = 3.93 \text{ kN}$]

Q5. A portion of the shifter mechanism for a manual car transmission is shown. For the 8 N force exerted on the shift knob, determine the corresponding force P exerted by the shift link BC on the transmission. Neglect friction in the ball-and-socket joint at O , in the joint at B , and in the slip tube near support D .



[ans: $P = 26.3 \text{ N}$]

1.4 Centroids and Distributed Forces

Forces are not always applied at a single point. Sometimes they are distributed over a surface or along a length. For example, the load of a train passing over a bridge can be simplified as an evenly distributed load.

We can simplify distributed loads by combining the total force and placing it at the centre of the distributed area.

Types of Distributed Loads

Common types of distributed loads include:

- Uniform distributed loads (constant intensity)
- Non-uniform distributed loads (varying intensity)

The resultant load R is represented by the area under the load intensity curve w (force per unit length). The resultant passes through the centroid of this area.

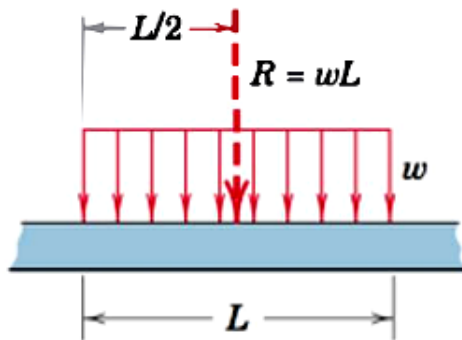


Figure 1.5: Uniformly distributed load.

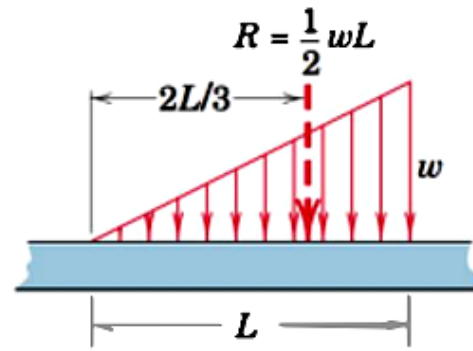


Figure 1.6: Non-uniform (triangular) distributed load.

Centroid of Common Areas

- Rectangle: Centroid at the centre
- Triangle: Centroid at one-third of the base from the larger end

If a load distribution consists of both a rectangle and a triangle, treat them as two separate resultant forces, each acting at its own centroid.

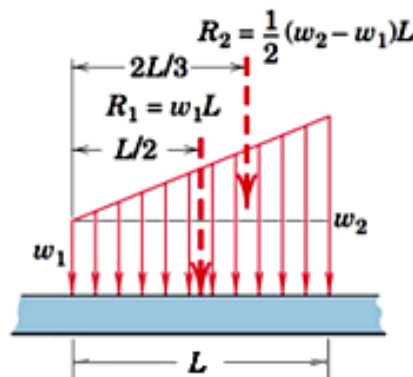
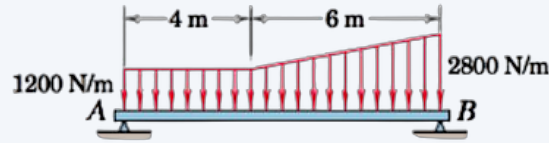


Figure 1.7: Combined rectangular and triangular load distribution.

Example 1.4 Finding Reactions for a Distributed Load

Problem: Determine the concentrated loads and external reactions for the simply supported beam subjected to the distributed load shown.

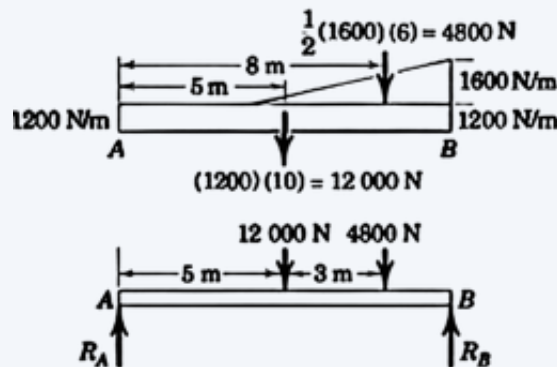
**Solution:**

Step 1: Divide the load distribution into rectangular and triangular areas.

Rectangular area: $R_1 = 1200 \times 4 = 4800 \text{ N}$ (acting at centre of the 4 m length)

Triangular area: $R_2 = \frac{1}{2}(1200 \times 6) = 3600 \text{ N}$ (acting at $\frac{2}{3}$ of 6 m from the left)

Step 2: Draw the free body diagram with the concentrated loads.



The loads act at their respective centroids.

Step 3: Write the equations of equilibrium.

$$\Sigma M_A = 0 = -4800(2) - 3600(4 + 4) + B_y(10)$$

$$\Sigma F_y = 0 = A_y + B_y - 4800 - 3600$$

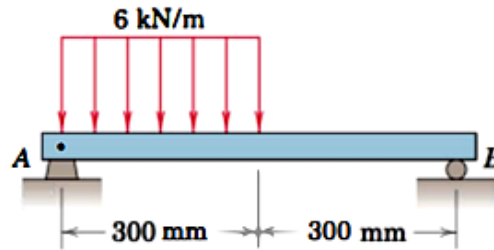
Step 4: Solve for the reactions.

$$B_y = 2400 \text{ N}$$

$$A_y = 6000 \text{ N}$$

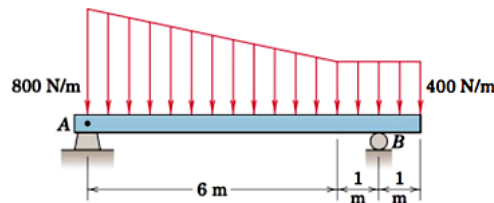
$A_y = 6.00 \text{ kN}, \quad B_y = 2.40 \text{ kN}$

Q1. Determine the reactions at A and B for the beam subjected to the uniform load distribution.



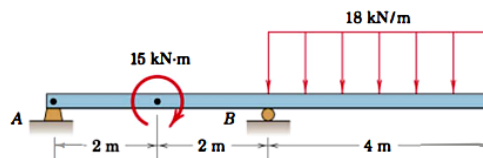
[ans: $R_A = 1.35$ kN up, $R_B = 0.45$ kN up]

Q2. Determine the reactions at the supports of the beam which is loaded as shown.



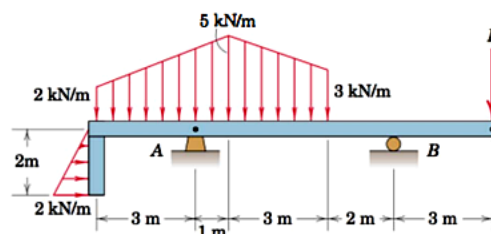
[ans: $R_A = 2230$ N up, $R_B = 2170$ N up]

Q3. Determine the reactions at A and B for the beam loaded as shown.



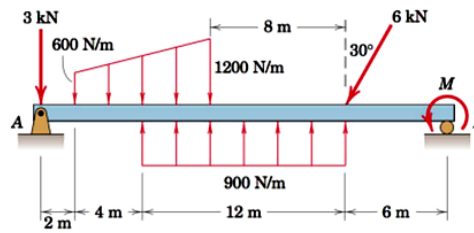
[ans: $R_A = 39.8$ kN down, $R_B = 111.8$ kN up]

Q4. For the beam and loading shown, determine the magnitude of the force F for which the vertical reactions at A and B are equal. With this value of F , calculate the magnitude of the pin reactions at A .



[ans: $F = 10.36$ kN, $R_A = 18.29$ kN]

Q5. Determine the magnitude of the moment M which will cause the beam to just begin to lift off the roller at B . For this value of M , determine the magnitude of the pin reaction at A .



[ans: $R_A = 5.49 \text{ kN}$, $M = 10.33 \text{ kN}\cdot\text{m}$]

Summary 1 Forces and Moments

Force A vector quantity representing the interaction between two bodies. It has both magnitude and direction.

$$\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j}, \quad F = \sqrt{F_x^2 + F_y^2}$$

Moment The rotational tendency of a force about a point.

$$M = Fd = Fd \sin \theta$$

Free Body Diagram A sketch of an isolated body showing all external forces and moments acting on it.

Equilibrium Equations For a body in static equilibrium:

$$\Sigma F_x = 0, \quad \Sigma F_y = 0, \quad \Sigma M = 0$$

Distributed Load A load spread over a length or area. The resultant is the area under the load intensity curve, acting at the centroid.

$$R = \int w(x) dx, \quad \bar{x} = \frac{\int xw(x) dx}{\int w(x) dx}$$

Chapter Checklist

- ☐ I can resolve forces into components and find resultant forces.
- ☐ I can calculate the moment of a force about a point.
- ☐ I can draw a free body diagram for a given system.
- ☐ I can identify the reactions at different types of supports.
- ☐ I can solve for unknown reactions using equilibrium equations.
- ☐ I can determine the resultant of a distributed load and find its location.

2. Internal Forces and Moment Diagrams

Internal forces and moments are crucial in structural engineering and mechanics, as they allow us to determine the loading acting within a member to ensure that the material can resist said loading. This chapter explores how to determine internal forces—normal force, shear force, and bending moment—and how to construct their corresponding diagrams.

2.1 Internal Forces

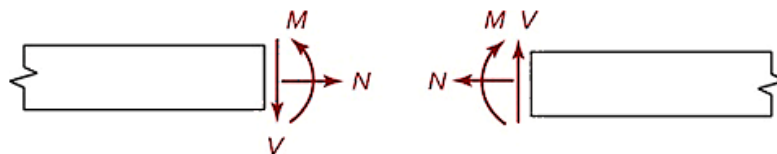
In statics, there are three types of loads that act on a system:

1. **External loads:** Concentrated forces, distributed loads, or concentrated moments applied to the system.
2. **Reaction forces or moments:** Reactions at supports caused by external loads or moments.
3. **Internal forces and moments:** Axial, shear, and/or bending forces within beams and other rigid bodies that are caused by external forces.

To analyse these internal loads, we cut through the beam to expose three internal forces:

- **M – Bending Moment:** Causes the beam to bend with curvature.
- **N – Normal Force:** Causes the beam to extend or compress along its length.
- **V – Shear Force:** Induces a tearing or shearing effect in the beam.

The following diagram illustrates how we can section a beam:

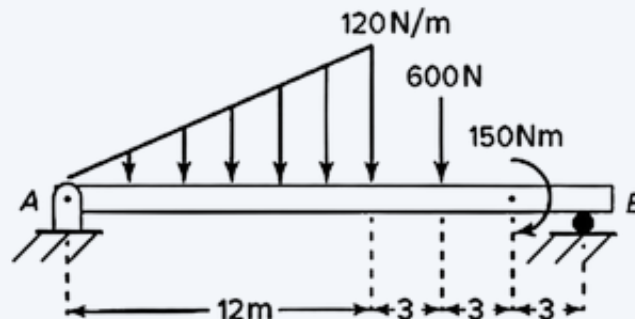


It is important to remember the correct positive sign conventions:

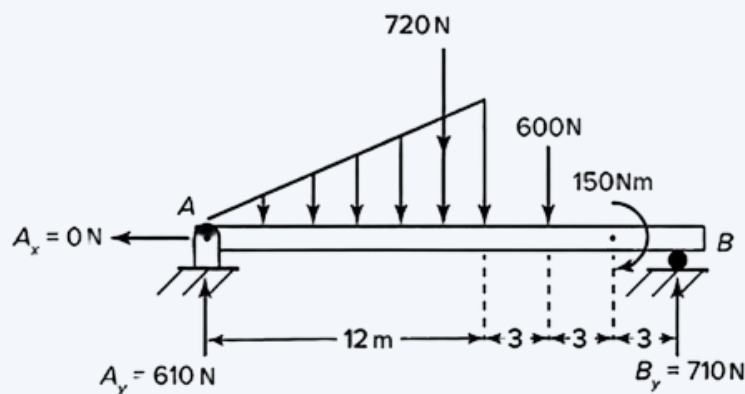
- Counter-clockwise moments are positive.
- Upward forces are positive.
- Rightward forces are positive.

Example 2.1 Finding Internal Forces in a Beam

Problem: For the given loaded beam, find the internal forces on the beam at a point just to the left of the 600 N concentrated load.

**Solution:**

- Step 1:** Find the global equilibrium. Draw a free body diagram of the beam and solve for the reactions.



$$\Sigma M_A = 0 = -720(8) - 600(15) - 150 + B_y(21)$$

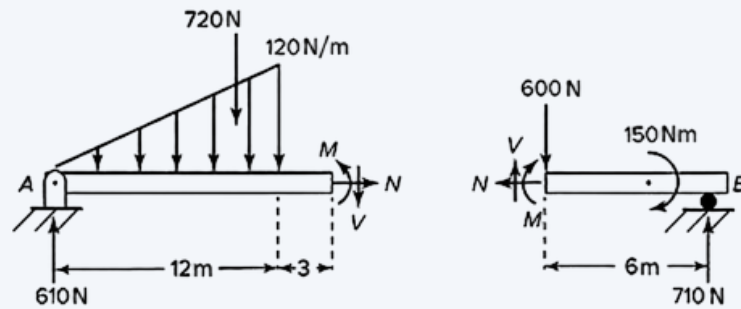
$$B_y = 710 \text{ N}$$

$$\Sigma F_x = 0 = A_x \implies A_x = 0$$

$$\Sigma F_y = 0 = B_y + A_y - 720 - 600 = 710 + A_y - 720 - 600$$

$$A_y = 610 \text{ N}$$

- Step 2:** Cut the beam at the point of interest and draw the free-body diagram of either side of the beam. Add M , N , and V to the diagram. If there is a concentrated force at the point of interest, cut the beam just slightly to the left or right.



3. **Step 3:** Use either free body diagram and solve for M , N , and V .

$$\Sigma F_x = 0 = N \implies N = 0$$

$$\Sigma F_y = 0 = 610 - V - 720 \implies V = -110 \text{ N}$$

$$\Sigma M_{\text{cut}} = 0 = M + 720(15 - 8) - 610(15) \implies M = 4110 \text{ Nm}$$

$$N = 0 \text{ N}, \quad V = -110 \text{ N}, \quad M = 4110 \text{ Nm}$$

2.2 Shear and Moment Diagrams: Graphic Method

Shear and moment diagrams provide a visual representation of the internal shear force and bending moment along the length of a beam. These diagrams are essential for identifying critical points where the maximum shear or moment occurs.

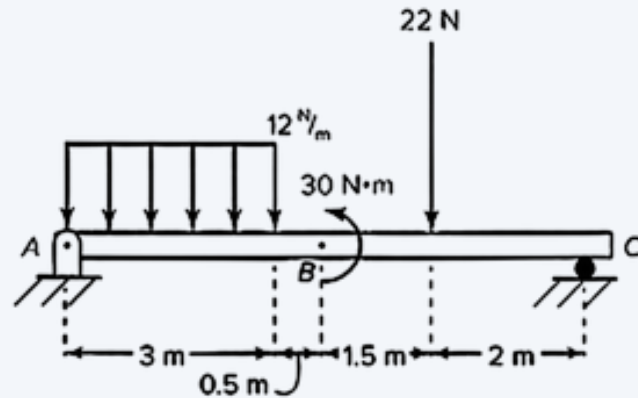
Graphic Method

The graphic method involves constructing the shear and moment diagrams using the relationships between load, shear, and moment. The key principles are:

- The slope of the shear diagram at any point equals the negative of the distributed load at that point.
- The slope of the moment diagram at any point equals the shear force at that point.
- The area under the shear diagram between two points equals the change in moment between those points.

Example 2.2 Drawing Shear and Moment Diagrams Using the Graphic Method

Problem: Draw the shear and moment diagrams for the following loaded beam.



Solution:

Step 1: Find the global equilibrium of the beam assuming both reaction forces are upwards.

$$\Sigma M_A = 0 = -12(3)(1.5) + 30 - 22(5) + C_y(7)$$

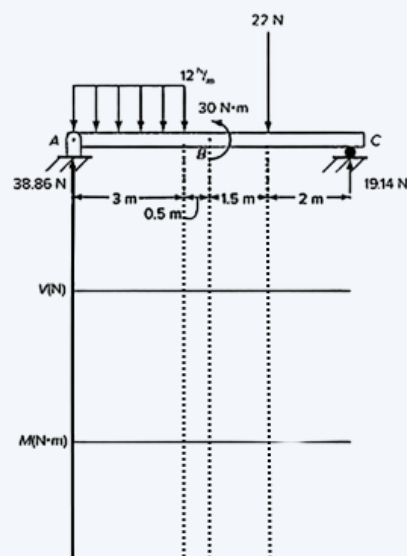
$$C_y = 19.14 \text{ N}$$

$$\Sigma F_y = 0 = C_y + A_y - 12(3) - 22 = 19.14 + A_y - 36 - 22$$

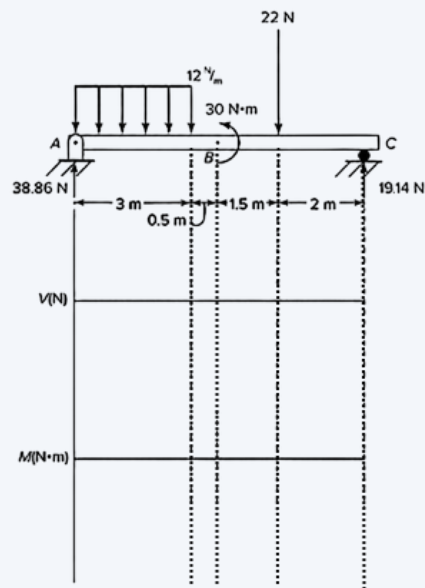
$$A_y = 38.86 \text{ N}$$

$$\Sigma F_x = 0 = A_x$$

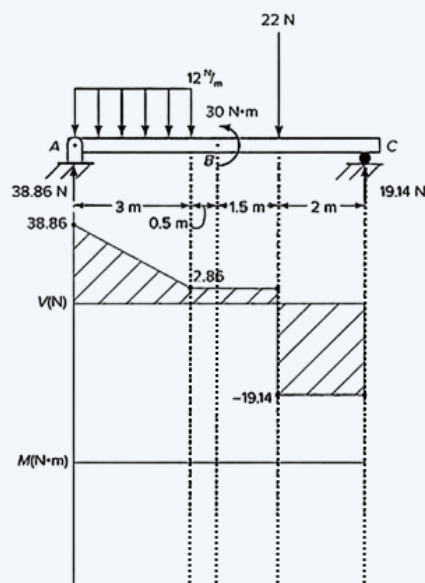
Step 2: Construct your axes.



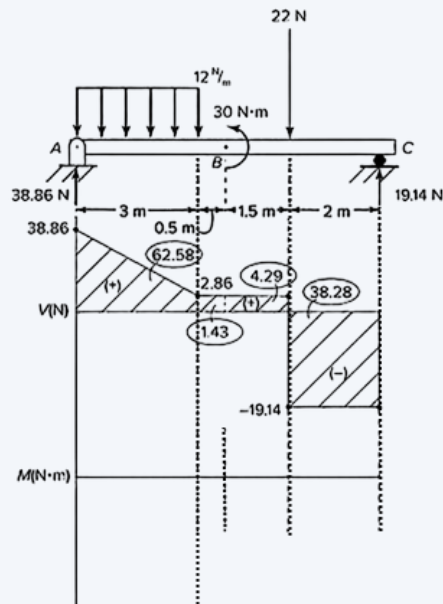
Step 3: Mark where the diagram discontinuities on the diagram.



Step 4: Graph the shear force diagram.



Step 5: Calculate the area of each shape on the V diagram.



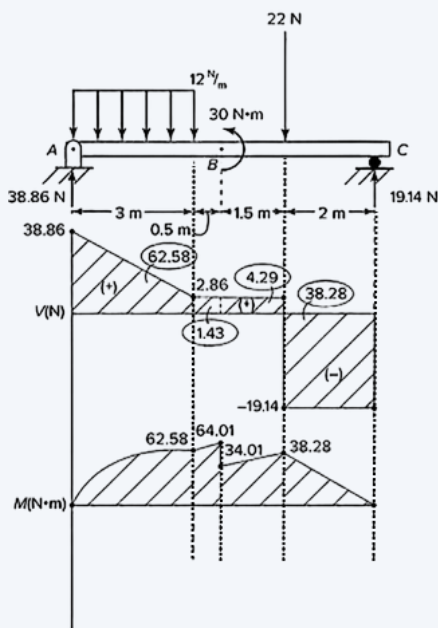
$$\text{Area 1} = \frac{1}{2}(3)(38.86 - 2.86) + (2.86)(3) = 62.58$$

$$\text{Area 2} = (2.86)(0.5) = 1.43$$

$$\text{Area 3} = (2.86)(1.5) = 4.29$$

$$\text{Area 4} = (19.14)(2) = 38.28$$

Step 6: Plot the M diagram from the areas derived in the previous step.



From these diagrams, you can easily identify where the maximum moment or shear force occurs.

2.3 Shear and Moment Diagrams: Equation Method

Sometimes the graphic method may not be sufficient to find the maximum moment or shear force, particularly when the loading results in a parabolic shear curve. In such cases, the equation method is the most effective approach.

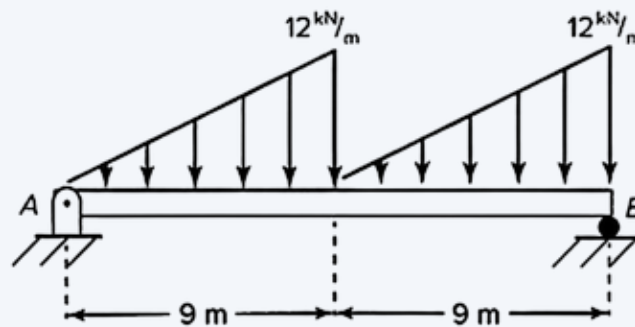
Equation Method

The equation method involves:

1. Determining the shear force and bending moment as functions of position x .
2. Finding the location where the shear force is zero to identify local maxima or minima in the moment diagram.
3. Substituting this location into the moment equation to find the maximum moment value.

Example 2.3 Drawing Shear and Moment Diagrams Using the Equation Method

Problem: Draw the shear and moment diagrams for the following loaded beam.



Solution:

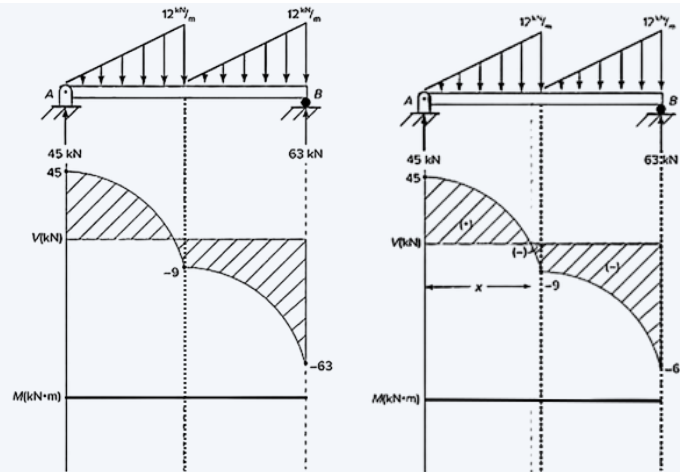
Step 1: Repeat the shear and moment diagram procedure to find reactions.

$$\Sigma M_A = 0 = -54(6) - 54(15) + B_y(18)$$

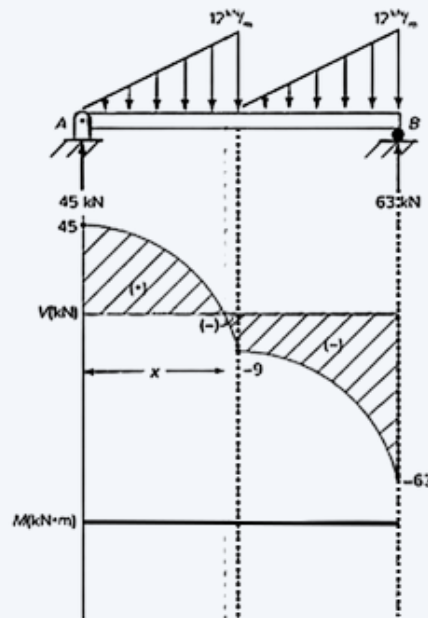
$$B_y = 63 \text{ kN}$$

$$\Sigma F_y = 0 = A_y + B_y - 54 - 54 = A_y + 63 - 54 - 54$$

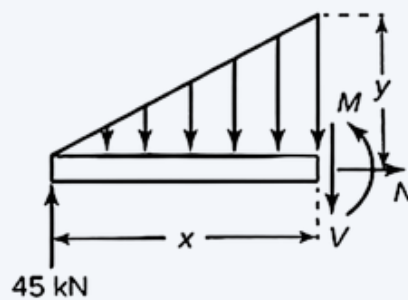
$$A_y = 45 \text{ kN}$$



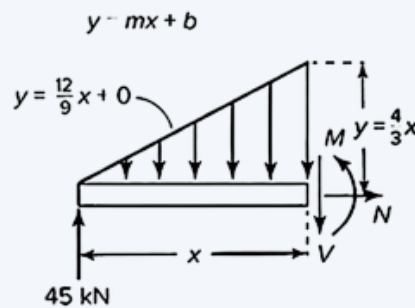
Step 2: When the shear force diagram goes from positive to negative and crosses the x -axis, create a vertical dashed line at this new location. This indicates a local maximum or minimum for the moment.



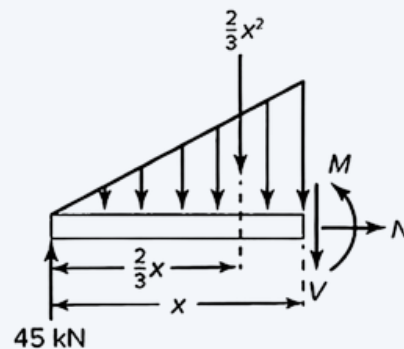
Step 3: Cut the original load curve at a distance x and draw the free-body diagram of the left side of the beam.



Step 4: Write the equation for the sloped line for the given load curve to find the height y of the curve at distance x .



Step 5: Convert the distributed load to a concentrated load as a function of x . Write equations for $\Sigma F_y = 0$ and $\Sigma M_x = 0$ and solve for V and M .



$$\text{Concentrated Load} = \frac{1}{2} \left(\frac{4}{3}x \right) (x) = \frac{2}{3}x^2$$

$$\Sigma F_y = 0 = 45 - V - \frac{2}{3}x^2 \implies V = 45 - \frac{2}{3}x^2$$

$$\Sigma M_x = 0 = \frac{2}{3}x^2 \left(x - \frac{2x}{3} \right) - 45(x) + M$$

$$= \frac{2}{3}x^2 \left(\frac{x}{3} \right) - 45(x) + M$$

$$M = 45x - \frac{2}{9}x^3$$

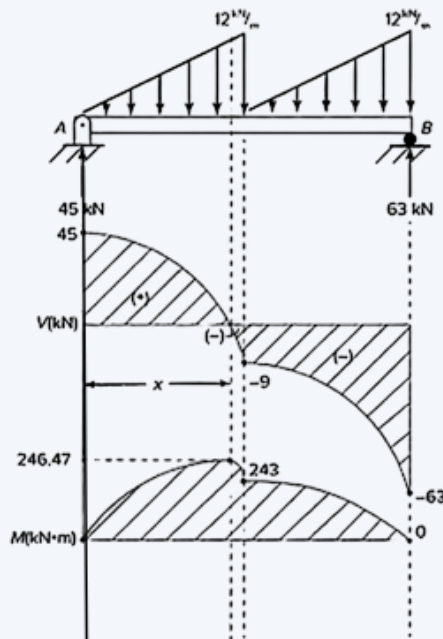
Step 6: Since the local maximum/minimum occurs when $V = 0$, set $V = 0$ and solve for x . Substitute into M to find the maximum moment.

$$0 = 45 - \frac{2}{3}x^2 \implies x = 8.22 \text{ m}$$

$$M_{\max} = 45(8.22) - \frac{2}{9}(8.22)^3 = 246.47 \text{ kNm}$$

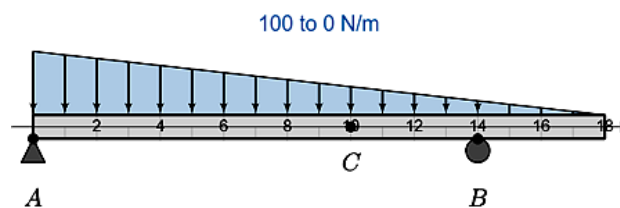
$$\text{Moment at endpoint: } M = 45(9) - \frac{2}{9}(9)^3 = 243 \text{ kNm}$$

Step 7: Complete the moment diagram using the derived values.



$$V(x) = 45 - \frac{2}{3}x^2, \quad M(x) = 45x - \frac{2}{9}x^3$$

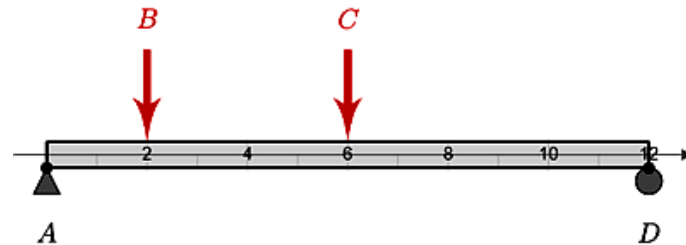
Q1. The 18 m beam shown supports a load that varies uniformly from 100 N/m at the left to 0 N/m at the right end. Point C is located at $x = 10$ m. Determine the reactions at pin A and roller B. Let the positive values indicate upwards force. Also determine the internal shear and bending moment at the section passing through point C. Use the standard convention for signs.



[ans: $A = 514.3 \text{ N}$, $B = 385.7 \text{ N}$, $V_C = -207.9 \text{ N}$, $M_C = 1069 \text{ Nm}$]

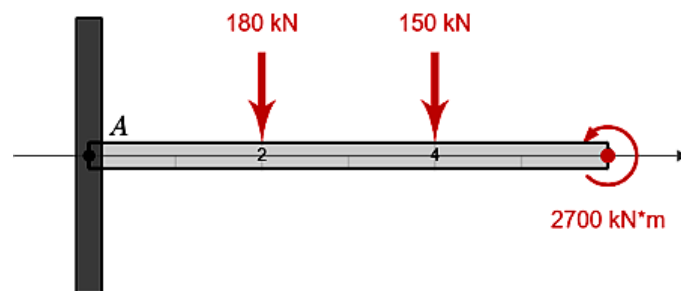
Q2. The 12 m long beam is loaded with a 750 kN force at B and a 1000 kN force at C, as shown. Work out the equations for the shear and bending moment functions for each of the three segments of the beam. Calculate this value for the ranges:

- (a) $0 \text{ m} < x < 2 \text{ m}$
- (b) $2 \text{ m} < x < 6 \text{ m}$
- (c) $6 \text{ m} < x < 12 \text{ m}$

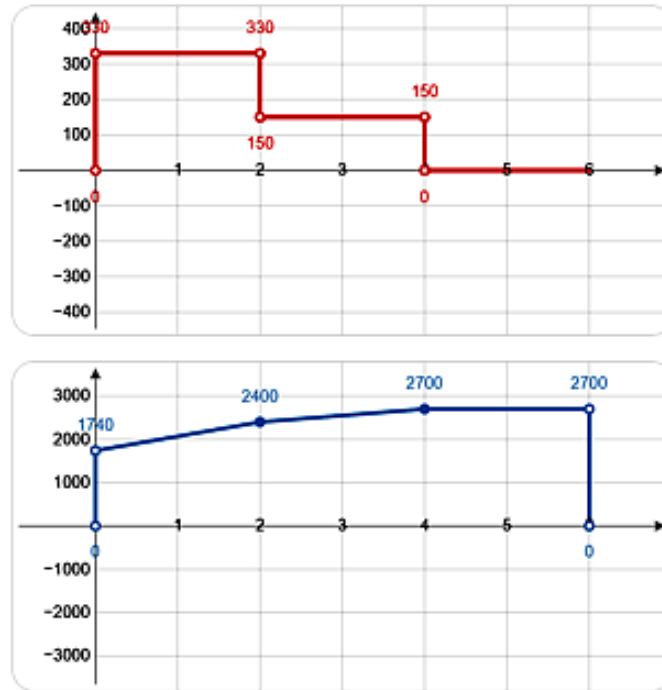


$$\left[\begin{array}{ll} V_a(x) = 1125 \text{ kN}, & M_a(x) = 1125x \text{ kNm} \\ \text{ans: } V_b(x) = 375 \text{ kN}, & M_b(x) = 1125x - 750(x - 2) \text{ kNm} \\ V_c(x) = -625 \text{ kN}, & M_c(x) = 1125x - 750(x - 2) - 1000(x - 6) \text{ kNm} \end{array} \right]$$

Q3. Draw the diagram and determine the reaction forces and maximum shear and bending moment using the correct sign conventions, and draw the shear and bending moment diagrams.

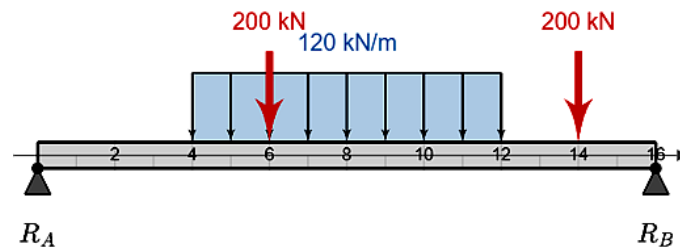


Shear and Moment Diagrams:

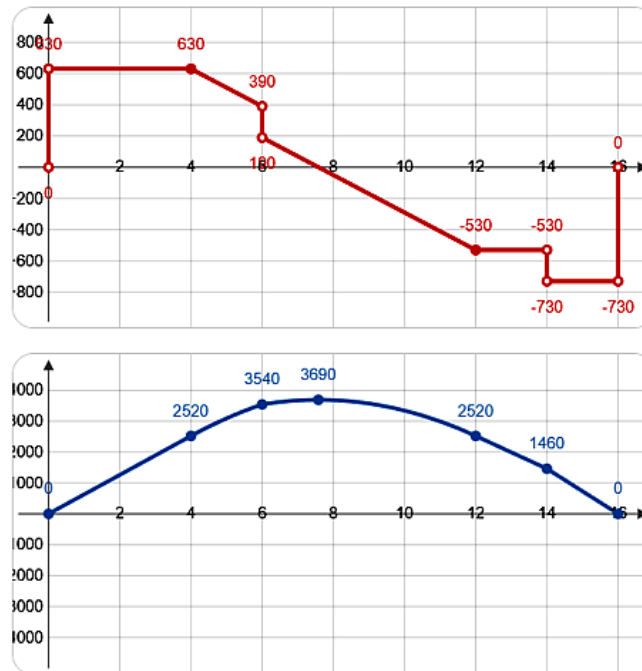


[ans: $A = 330 \text{ kN}$, $M_A = -1740 \text{ kNm}$, $V_{\max} = 330 \text{ kN}$, $M_{\max} = 2700 \text{ kNm}$]

Q4. Work out the reaction forces and draw the shear and moment diagrams for the beam and loading shown, and determine the maximum shear and moment.

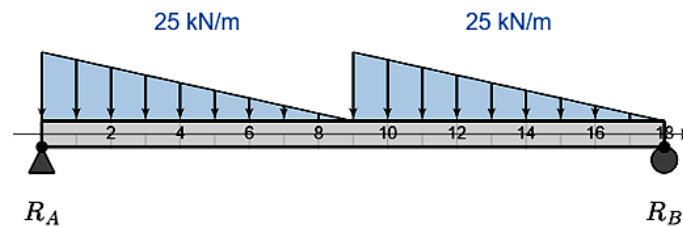


Shear and Moment Diagrams:

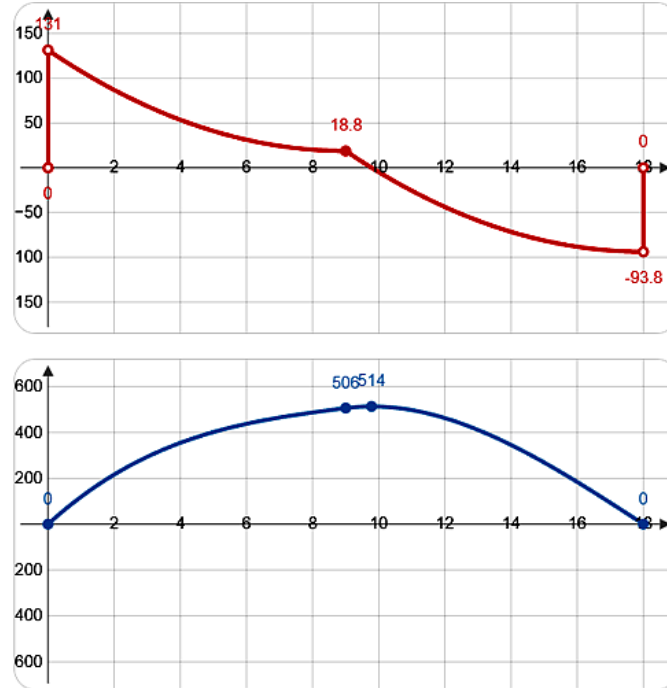


[ans: $A = 630$ kN, 730 kN, $V_{\max} = -730$ kN, $M_{\max} = 3690$ kNm]

Q5. Work out the reaction forces and draw the shear and moment diagrams for the beam and loading shown. Then, determine the maximum shear and moment.



Shear and Moment Diagrams:



[ans: $A = 131.3 \text{ kN}$, $B = 93.75 \text{ kN}$, $V_{\max} = 131.3 \text{ kN}$, $M_{\max} = 513.5 \text{ kNm}$]

2.4 Chapter Review

Summary 2 Internal Forces and Moment Diagrams

Internal Forces

When a beam is sectioned, three internal forces are revealed:

- **N (Normal Force):** Acts along the axis of the member, causing tension or compression. Positive normal force indicates tension; negative indicates compression.
- **V (Shear Force):** Acts perpendicular to the axis, causing sliding or shearing.
- **M (Bending Moment):** Causes bending of the member. Positive bending moment causes sagging (concave up).

Shear and Moment Diagrams

- The **shear diagram** shows how the internal shear force varies along the length of the beam. The slope of the shear diagram equals the negative of the distributed load.
- The **moment diagram** shows how the internal bending moment varies along the length. The slope of the moment diagram equals the shear force.
- The area under the shear diagram between two points equals the change in moment between those points.
- Points where the shear force crosses zero correspond to local maxima or minima in the moment diagram.

Graphic Method

1. Determine reactions using equilibrium equations.
2. Construct axes with the beam length on the horizontal axis.
3. Mark points of discontinuity (concentrated loads, supports, etc.).
4. Draw the shear diagram using the relationships between load and shear.
5. Calculate areas under the shear diagram.
6. Plot the moment diagram using the areas.

Equation Method

1. Determine reactions using equilibrium equations.
2. Write expressions for shear and moment as functions of x for each segment.
3. Find where $V = 0$ to locate maximum/minimum moments.
4. Substitute into the moment equation to find extreme values.

Sign Conventions

- Counter-clockwise moments are positive.
- Upward forces are positive.
- Rightward forces are positive.

Chapter Checklist

- ☐ I can determine internal forces (normal, shear, and bending moment) by sectioning a beam.
 - ☐ I can construct shear and moment diagrams using the graphic method.
 - ☐ I can construct shear and moment diagrams using the equation method.
 - ☐ I can identify maximum shear and moment values from diagrams.
 - ☐ I can write shear and moment functions for each segment of a beam.
- 



3. Second Moment of Area

Stress and strain are fundamental concepts in mechanics of materials. Stress describes the distribution of internal forces within a body, while strain measures the deformation that occurs as a result of applied loads. This chapter covers axial stress, shear stress, strain, centroids, and the second moment of area, which are essential for understanding how structures respond to loading.

3.1 Tensile Stress and Strain

3.1.1 Stress

Stress can be defined as the quantity that describes the distribution of internal forces within a body. Mathematically, stress is defined as:

$$\sigma = \frac{F}{A}$$

where F is the applied force and A is the cross-sectional area. The unit of stress is N/m^2 , also known as the pascal (Pa). In engineering applications, stress is typically measured in MPa or N/mm^2 .

If tension is applied, the stress elements will elongate. Conversely, if compression is applied, the element contracts. Stress can be considered at an infinitesimally small cube at the point of interest.

3.1.2 Strain

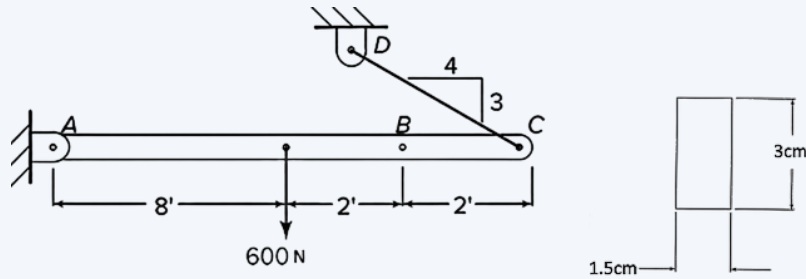
Engineering strain is the ratio between change in dimension and its initial dimension. The units of strain are dimensionless, as it is a measure of deformation.

$$\varepsilon = \frac{\Delta l}{l}$$

Normal strain, also referred to as axial strain, occurs when a body is loaded in the axial direction. For most materials, tensile forces yield positive strain, while compressive forces yield negative strain.

Example 3.1 Finding Normal Stress

Problem: Find the normal stress at point B. The cross-section is 1.5 cm by 3 cm as shown.



Solution:

Step 1: Find the global equilibrium.

$$\Sigma M_A = 0 = \frac{3}{5}F_{CD}(12) - 600(8)$$

$$F_{CD}(0.6)(12) = 4800$$

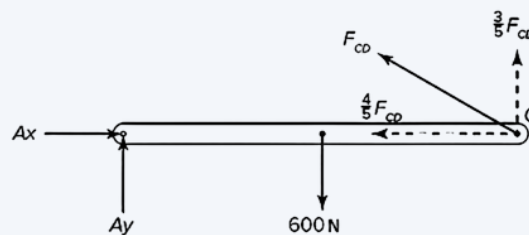
$$F_{CD} = 666.7 \text{ N}$$

$$\Sigma F_x = 0 = A_x - \frac{4}{5}F_{CD}$$

$$A_x = 0.8(666.7) = 533.3 \text{ N}$$

$$\Sigma F_y = 0 = A_y - 600 + \frac{3}{5}F_{CD}$$

$$A_y = 600 - 0.6(666.7) = 200 \text{ N}$$



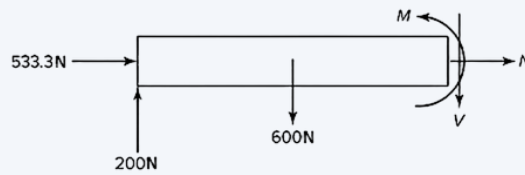
Step 2: Cut the beam through point B and draw the free body diagram.

Consider the right side of the cut.

$$\Sigma F_x = 0 = 533.3 + N$$

$$N = -533.3 \text{ N}$$

The negative sign indicates compression.



Step 3: Calculate the stress.

$$\sigma = \frac{N}{A} = \frac{-533.3}{(3)(1.5)} = -118.5 \text{ N/cm}^2$$

$$\sigma_B = -118.5 \text{ N/cm}^2 \text{ (compression)}$$

3.2 Shear Stress and Strain

3.2.1 Shear Stress

Shear stress results from a perpendicular force that can be thought of as a "tearing force." There is an upwards force on one side of the "tear" and a downward force on the other side. Shear stress is defined as a shear force divided by the initial (parallel) area:

$$\tau = \frac{F_s}{A_0}$$

Shear stress normally generates a change in shape that is measured as shear strain.

3.2.2 Shear Strain

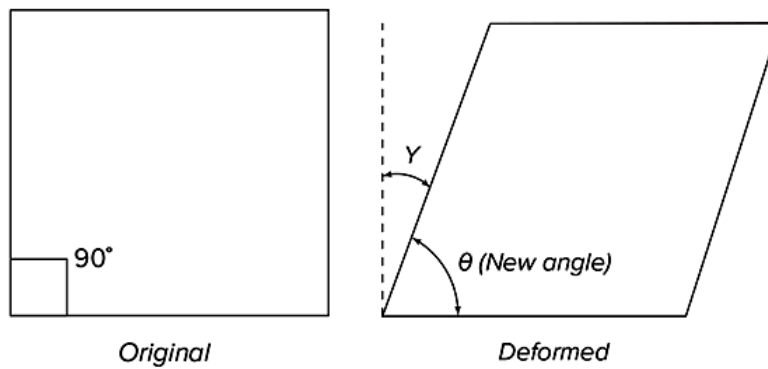
Shear strain corresponds to the change of angle under deformation from shearing forces. It is expressed as the Greek letter gamma (γ). The units for shear strain are in radians and are dimensionless.

$$\gamma = \gamma_{xy} = 2\epsilon_{xy} = \frac{\Delta x}{y} = \tan \theta$$

where

$$\epsilon_{xy} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

For small angles, $\tan \theta \approx \theta$ using the small-angle approximation.

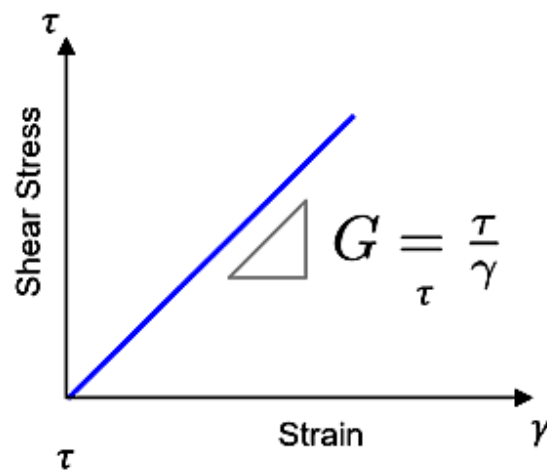


3.2.3 Shear Modulus

Much like a regular stress-strain diagram, there is a shear stress-strain diagram. The linear portion of the graph leads to the Shear Modulus of Elasticity (G):

$$G = \frac{\tau}{\gamma}$$

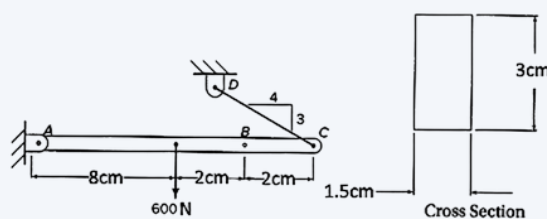
The units are typically MPa.



please double check this figure - some redundant symbols

Example 3.2 Shear Stress in a Pin

Problem: If the maximum allowable shear stress of the pin at point A is 5000 N/cm², find the minimum diameter of the pin at point A.



Solution:

Step 1: From previous equilibrium analysis,

$$A_x = 533.3 \text{ N (to the right)}, \quad A_y = 200 \text{ N (upward)}$$

Step 2: Find the shear force (V) acting on the pin at A.

$$V = \sqrt{533.3^2 + 200^2} = 569.6 \text{ N}$$

Step 3: Solve for the diameter using the double shear equation.

$$\tau = \frac{V}{2A}$$

$$5000 = \frac{569.6}{2(\pi r^2)}$$

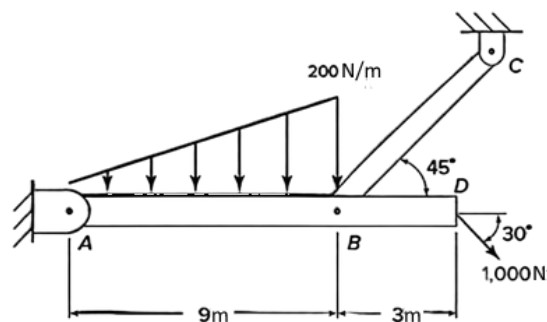
$$r^2 = \frac{569.6}{2\pi(5000)} = 0.0181$$

$$r = 0.135 \text{ cm}$$

$$d = 0.27 \text{ cm}$$

Q1. For the frame shown, determine:

- The support reactions at A and B.
- The moment M , shear V , and normal force N in the horizontal member at a location 4 m to the right of pin A.



[ans: $A_x = -2132.7 \text{ N}$, $A_y = 133.36 \text{ N}$, $BC = 1791.3 \text{ N}$, $N = 2132.7 \text{ N}$, $V = -44.42 \text{ N}$, $M = 296.99 \text{ N/m}$]

3.3 Centroids

A centroid is a location in space $(\bar{x}, \bar{y}, \bar{z})$ which balances the weight, mass, and area at a given point. Finding the centre of mass helps simplify the study of dynamic responses from bodies. The formulas for centroids are:

3.3.1 Centre of Mass

$$\bar{x} = \frac{\sum x_i M_i}{\sum M_i}, \quad \bar{y} = \frac{\sum y_i M_i}{\sum M_i}, \quad \bar{z} = \frac{\sum z_i M_i}{\sum M_i}$$

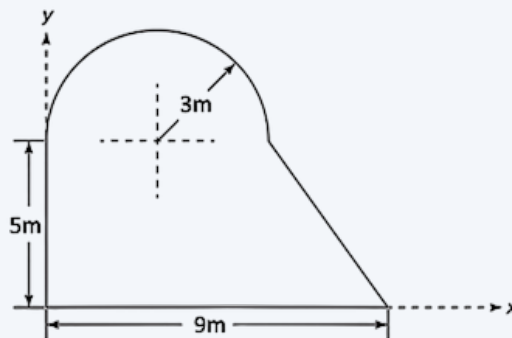
For volume and area, replace M with V or A . The subscript i represents a particular piece of the object being analysed.

3.3.2 Composite Shape Method

The composite shape method involves dividing an irregular shape into simpler components, calculating the centroid of each component, and then using the weighted average formula.

Example 3.3 Finding the Centroid of a Composite Shape

Problem: Find $\bar{x}$ and $\bar{y}$ for the given shape as measured from the coordinate axes.

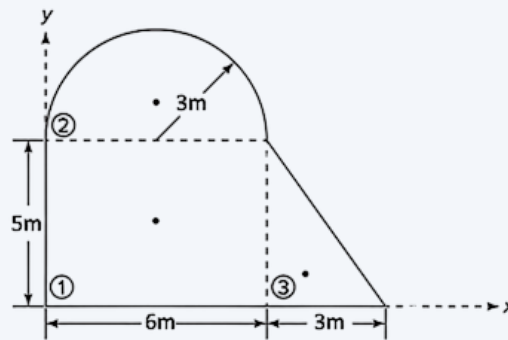


Solution:

Step 1: Identify the relevant composite equation.

$$\bar{x} = \frac{\sum x_i A_i}{\sum A_i}, \quad \bar{y} = \frac{\sum y_i A_i}{\sum A_i}$$

Step 2: Divide the composite shape into components: rectangle, semi-circle, and triangle.



Step 3: Expand the centroid equations.

$$\bar{x} = \frac{\bar{x}_1 A_1 + \bar{x}_2 A_2 + \bar{x}_3 A_3}{A_1 + A_2 + A_3}$$

$$\bar{y} = \frac{\bar{y}_1 A_1 + \bar{y}_2 A_2 + \bar{y}_3 A_3}{A_1 + A_2 + A_3}$$

Step 4: Substitute the values.

$$\bar{x} = \frac{(3)(30) + (3) \left(\frac{\pi(3)^2}{2} \right) + (7) \left(\frac{1}{2} \right) (3)(5)}{30 + \frac{\pi(3)^2}{2} + \frac{1}{2}(15)}$$

$$\bar{y} = \frac{(2.5)(30) + \left(5 + \frac{4(3)}{3\pi} \right) \left(\frac{\pi(3)^2}{2} \right) + \left(\frac{5}{3} \right) \left(\frac{1}{2} \right) (3)(5)}{30 + \frac{\pi(3)^2}{2} + \frac{1}{2}(15)}$$

Step 5: Solve.

$$\bar{x} = \frac{90 + 42.41 + 52.5}{30 + 14.14 + 7.5} = \frac{184.9}{51.64} = 3.58 \text{ m}$$

$$\bar{y} = \frac{75 + 88.69 + 12.5}{30 + 14.14 + 7.5} = \frac{176.19}{51.64} = 3.41 \text{ m}$$

$$\boxed{\bar{x} = 3.58 \text{ m}, \quad \bar{y} = 3.41 \text{ m}}$$

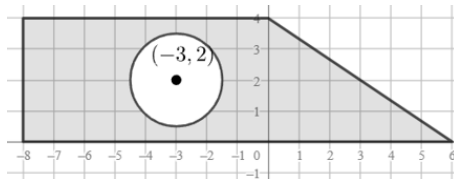
Alternative Method: Table Method

Area	x_i	y_i	A_i	$x_i A_i$	$y_i A_i$
1	3	2.5	30	90	75
2	3	6.27	14.14	42.42	88.66
3	7	1.67	7.5	52.5	12.53
Σ			51.64	184.9	176.2

$$\bar{x} = \frac{\sum x_i A_i}{\sum A_i} = \frac{184.9}{51.64} = 3.58 \text{ m}$$

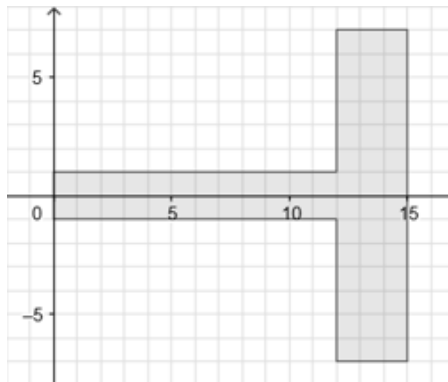
$$\bar{y} = \frac{\sum y_i A_i}{\sum A_i} = \frac{176.2}{51.64} = 3.41 \text{ m}$$

Q1. 1. Determine the centroid of the composite shape shown.



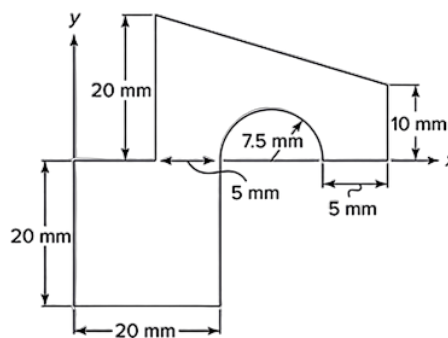
[ans: $\bar{x} = -2.24 \text{ cm}$, $\bar{y} = 1.78 \text{ cm}$]

Q2. 2. Determine the centroid of the composite shape shown below.



[ans: $\bar{x} = 10.77 \text{ cm}$, $\bar{y} = 0 \text{ cm}$]

Q3. 3. Determine the centroid of the composite shape shown.

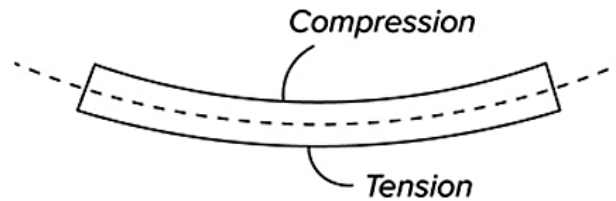


[ans: $\bar{x} = 16.55 \text{ mm}$, $\bar{y} = -1.99 \text{ mm}$]

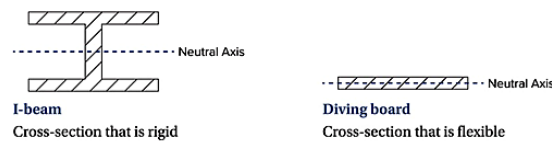
3.4 Second Moment of Area

The second moment of area is the resistance factor to bending. It tells us how the area of a section is distributed about a particular axis. An alternative name for this is the **moment of inertia**. The second moment of area is also used to calculate bending stress.

When a beam bends, the top section is in compression and the bottom section is in tension. At the neutral axis, the stress is zero. For straight beams, the neutral axis coincides with the centroidal axis.

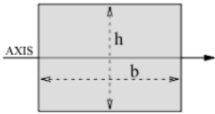
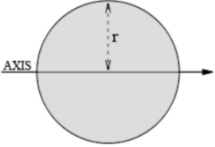
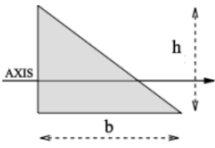


The more area that is further from the neutral axis, the less the cross-section will bend. A diving board is flexible because the neutral axis is located close to both the top and bottom. An I-shaped beam has a large distance between the neutral axis and both the top and bottom, making it more stiff.



3.4.1 Common Shapes

The following table summarises the properties of common shapes.

Shape	Diagram	Second Moment of Area	Centroid
Rectangle		$I_c = \frac{bh^3}{12}$	Centre
Circle		$I_c = \frac{\pi d^4}{64}$	Centre
Triangle		$I_0 = \frac{bh^3}{36}$	$\bar{x} = \frac{h}{3}, \bar{y} = \frac{b}{3}$

Note: I_c can only be used if the piece's centroid lies on the neutral axis.

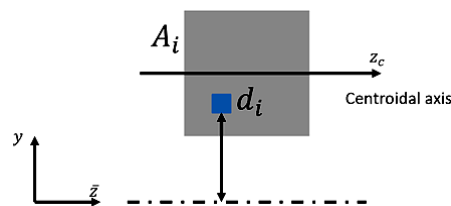
Notation:

- I_{x_c} and I_{y_c} : second moment of area of an individual piece about its centroid
- I_x and I_y : second moment of area of the entire composite shape about its centroid
- I'_x and I'_y : second moment of area for an axis that is not the centroid

3.5 Parallel Axis Theorem

The parallel axis theorem can be used to find the second moment of area about any axis that is parallel to the axis passing through the centroid.

$$I_z = I_{z_c} + d^2 A$$

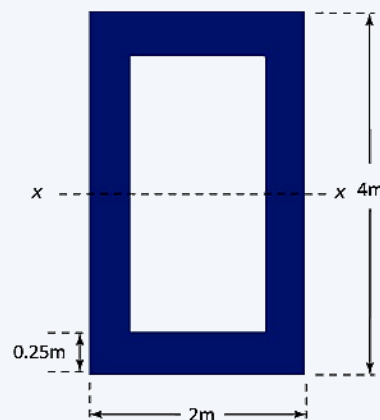


To find the second moment of area of a composite shape:

1. Break the composite shape into simpler shapes.
2. Calculate the centroid using $\bar{x} = \frac{\sum x_i A_i}{\sum A_i}$ and $\bar{y} = \frac{\sum y_i A_i}{\sum A_i}$.
3. Use $I_x = \sum (I_{x_c} + A d^2)$.

Example 3.4 Second Moment of Area (Without Parallel Axis Theorem)

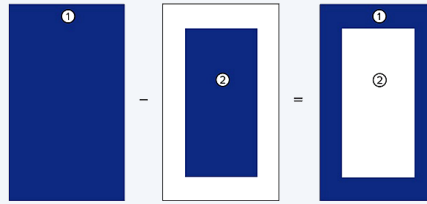
Problem: Find I_x about the centroidal axis for the whole part.



Solution:

Step 1: The centroid has already been determined to be 2 cm from the bottom.

Step 2: Break the composite shape into simpler shapes.



Step 3: Calculate the second moment of area.

$$I_x = \Sigma(I_{x_c} + Ad^2)$$

Here, $d = 0$ for both inner and outer rectangles as each centroid lies on the neutral axis.

$$I_x = I_{x_c \text{ outer}} - I_{x_c \text{ inner}}$$

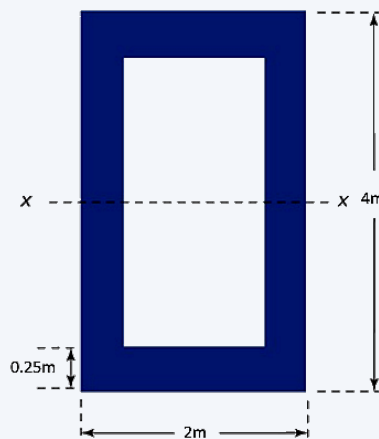
$$I_x = \frac{1}{12}(2)(4)^3 - \frac{1}{12}(1.5)(3.5)^3$$

$$I_x = 10.67 - 5.36 = 5.31 \text{ cm}^4$$

$$I_x = 5.31 \text{ cm}^4$$

Example 3.5 Second Moment of Area (With Parallel Axis Theorem)

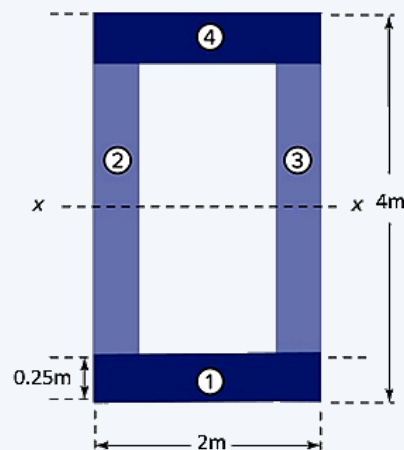
Problem: Find I_x about the centroidal axis.



Solution:

Step 1: The centroid has already been determined to be 2 cm from the bottom.

Step 2: Divide the shape into four rectangles.



Step 3: Apply the parallel axis theorem.

$$I_x = I_{x_1} + I_{x_2} + I_{x_3} + I_{x_4}$$

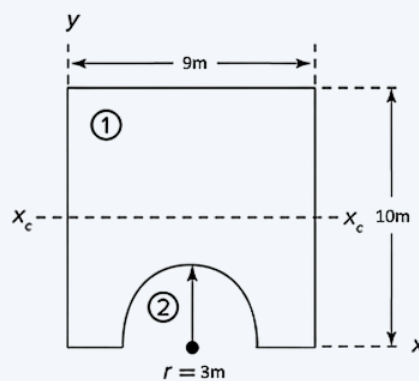
$$I_x = \left[\frac{1}{12}(2)(0.25)^3 + (0.5)(1.875)^2 \right] + \left[\frac{1}{12}(0.25)(3.5)^3 + (0.875)(0)^2 \right] \\ + \left[\frac{1}{12}(0.25)(3.5)^3 + (0.875)(0)^2 \right] + \left[\frac{1}{12}(2)(0.25)^3 + (0.5)(1.875)^2 \right]$$

$$I_x = 1.76 + 0.89 + 0.89 + 1.76 = 5.31 \text{ cm}^4$$

$$I_x = 5.31 \text{ cm}^4$$

Example 3.6 Composite Shape with a Centroid

Problem: Find I_x about the centroidal axis for the given shape.



Solution:

Step 1: Determine the location of the neutral axis by finding $\bar{y}$.

y	A	yA
5	90	450
1.27	-14.14	-18
	$\Sigma A = 75.86$	$\Sigma yA = 432$

$$\bar{y} = \frac{432}{75.86} = 5.7 \text{ cm}$$

Step 2: Find I_x using the parallel axis theorem.

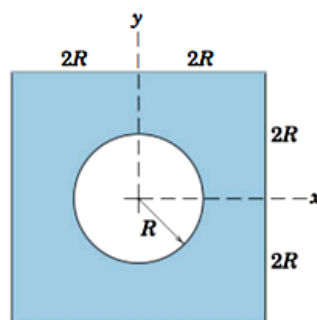
$$I_x = \Sigma(I_{x_c} + Ad^2)$$

$$I_x = \left[\frac{1}{12}(9)(10)^3 + (90)(0.7)^2 \right] - \left[\frac{\pi}{8}(3)^4 - \frac{8(3)^4}{9\pi} + (14.14)(4.43)^2 \right]$$

$$I_x = 794.1 - 286.4 = 507.7 \text{ cm}^4$$

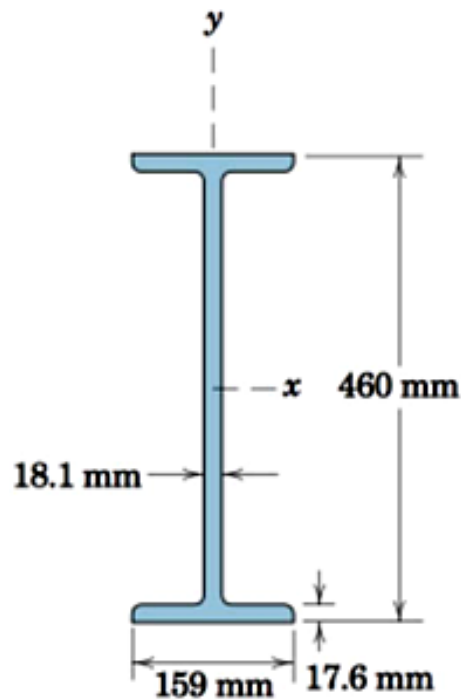
$$I_x = 507.7 \text{ cm}^4$$

Q1. 1. Determine the second moment of area about the x -axis of the square area with the central circular hole.



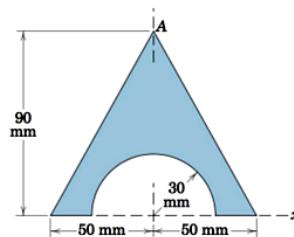
$$[\text{ans: } I_x = 20.5R^4]$$

Q2. 2. Determine the second moment of area of the I-shaped beam about the x - and y -axes.



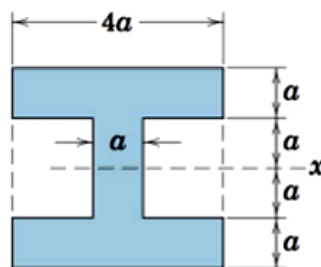
$$[\text{ans: } I_x = 3.9 \times 10^8 \text{ mm}^4, I_y = 1.2 \times 10^7 \text{ mm}^4]$$

Q3. 3. Calculate the second moment of area of the shaded area about the x -axis.



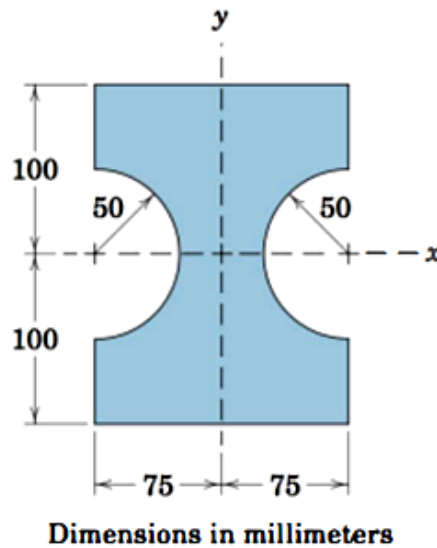
$$[\text{ans: } 5.76 \times 10^6 \text{ mm}^4]$$

Q4. 4. Determine the second moment of area of the shaded area about the x -axis.



$$[\text{ans: } I_x = \frac{58a^4}{3}]$$

Q5. 5. Determine the second moment of area of the shaded area about both the x - and y -axes.



[ans: $I_x = 95.1 \times 10^6 \text{ mm}^4$, $I_y = 32.2 \times 10^6 \text{ mm}^4$]

Summary 3 Stress, Strain and Cross-Sectional Properties

Stress The internal force per unit area within a body.

$$\sigma = \frac{F}{A} \quad (\text{normal stress}), \quad \tau = \frac{F_s}{A_0} \quad (\text{shear stress})$$

Strain The measure of deformation of a material.

$$\varepsilon = \frac{\Delta l}{l} \quad (\text{normal strain}), \quad \gamma = \tan \theta \quad (\text{shear strain})$$

Centroid The geometric centre of an area.

$$\bar{x} = \frac{\sum x_i A_i}{\sum A_i}, \quad \bar{y} = \frac{\sum y_i A_i}{\sum A_i}$$

Second Moment of Area A measure of a cross-section's resistance to bending.

$$I = \int y^2 dA$$

Parallel Axis Theorem Relates the second moment of area about any axis to that about a parallel axis through the centroid.

$$I = I_c + Ad^2$$

Chapter Checklist

- ☐ I can calculate normal stress and strain in a member under axial loading.
- ☐ I can calculate shear stress and shear strain.

- ☐ I can find the centroid of composite shapes using the table method.
- ☐ I can calculate the second moment of area for simple and composite shapes.
- ☐ I can apply the parallel axis theorem to find second moments of area about any axis.
- ☐ I can solve problems involving pin shear and bearing stress.



4. Beam Bending

In this chapter, we continue the study of beams under load by determining the relationship between the bending moment and the deflection it causes, and how to calculate the distribution of normal stress and the magnitude of deflection. We also examine the shear stresses that arise from shear forces, which act in addition to the normal stresses from bending.

4.1 Normal Stress due to Bending Moment

In Chapter 2, we covered the behaviour of internal forces for beams under load, with one of the components being the bending moment. When under load, the beam bends due to the bending moment that is created. The material resists this deformation with internal forces developing within the beam over an area, creating stress. To understand the distribution of stress, we can imagine the beam to be many layers of individual fibres, as depicted in Figure 4.1 below. We can observe that the fibres at the top have been compressed, so there is negative (compressive) stress, while the fibres at the bottom have elongated, so there is positive (tensile) stress. However, some fibres remain the same length; this is called the neutral axis.

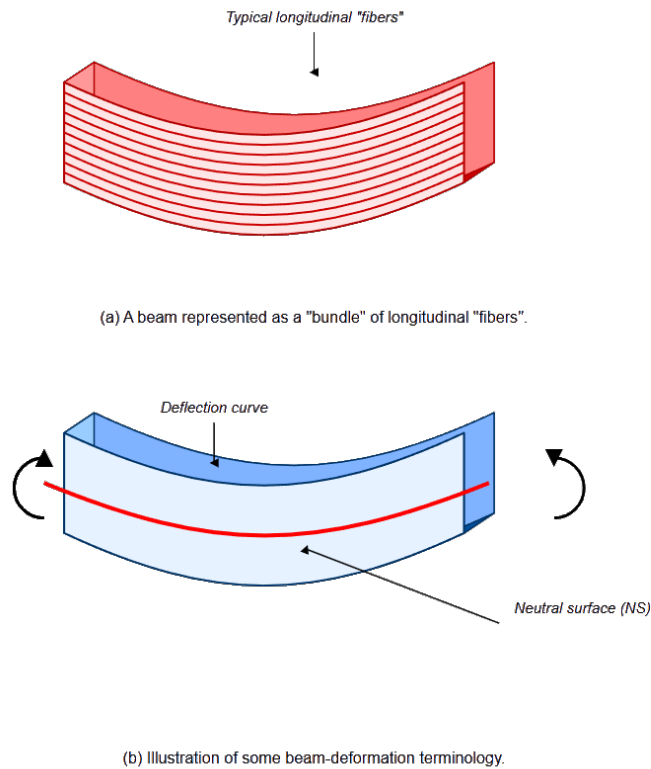


Figure 4.1: Visualisation of normal stress distribution in a beam under bending

The formula relating the normal stress and bending moment is:

$$\sigma = \frac{My}{I}$$

where:

- σ = normal stress (Pa or N/m²)
- M = bending moment (N·m)
- y = distance from the neutral axis (m)
- I = second moment of area (m⁴)

From the formula, we see that at the neutral axis, $y = 0$, so the normal stress at that point is zero. To determine the highest compressive stress, we take the distance furthest above the centroid; to find the highest tensile stress, we take the distance furthest below the centroid. However, this order reverses if the bending moment is negative.

Calculating bending stress involves determining:

- Centroids
- Moment diagrams
- Second moment of area

When calculating the maximum stress, the distance from the neutral axis to the surface can be denoted as c . This value is often used to calculate the section modulus:

$$Z = \frac{I}{c}$$

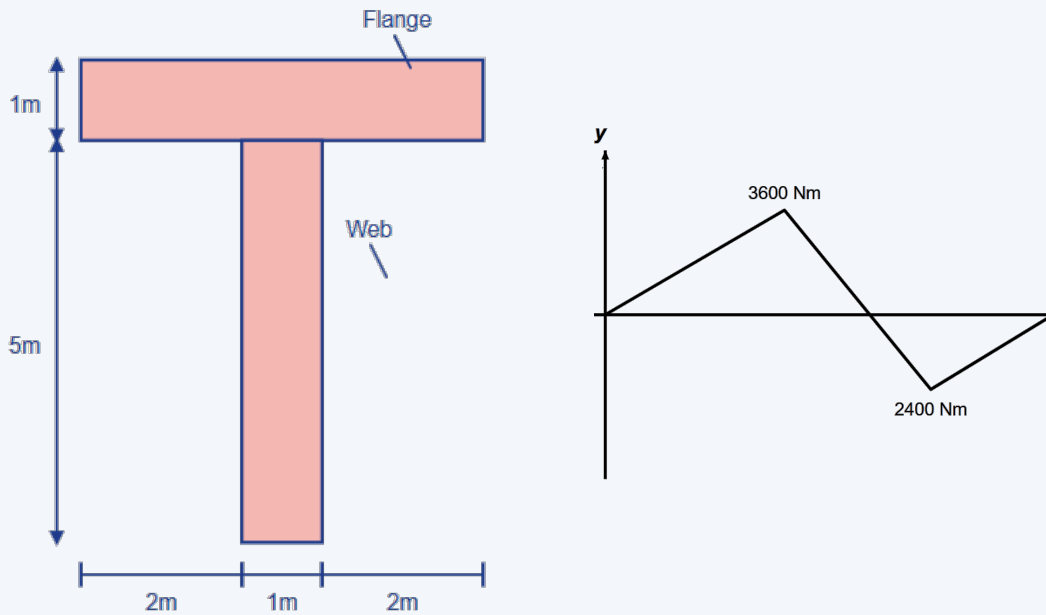
which shows a cross-section's resistance to bending. This can be substituted back

into the equation for maximum stress:

$$\sigma = \frac{M}{Z}$$

Example 4.1 Determining the Largest Compressive Stress in a T-Beam

Problem: A beam has a cross section consisting of a T-shape. The cross-section dimensions and moment diagram have been provided below. Determine the largest compressive stress in the beam.



Solution:

Step 1: Finding the Centroid

To find the centroid of the whole cross-section, we compute the centroid of each component, using the bottom of the shape as a reference.

$$y_{c1} = 0.5 + 5 = 5.5 \quad (\text{top flange}), \quad y_{c2} = 2.5 \quad (\text{web})$$

$$A_1 = 1 \times 5 = 5, \quad A_2 = 1 \times 5 = 5, \quad \sum A = 10$$

$$y_c = \frac{\sum y_i A_i}{\sum A} = \frac{(5.5)(5) + (2.5)(5)}{10} = \frac{40}{10} = 4 \text{ units}$$

Step 2: Finding the Second Moment of Area

We use the parallel axis theorem: $I = I_{\text{centroid}} + d^2 A$.

$$I_{x1} = \frac{5 \times 1^3}{12} + (1 + 0.5)^2 \times 5 = \frac{5}{12} + 1.5^2 \times 5 = 0.4167 + 11.25 = 11.6667$$

$$I_{x2} = \frac{1 \times 5^3}{12} + (1.5)^2 \times 5 = \frac{125}{12} + 2.25 \times 5 = 10.4167 + 11.25 = 21.6667$$

$$I_{\text{total}} = I_{x1} + I_{x2} = 33.33 \text{ (units}^4\text{)}$$

Step 3: Finding the Maximum Compressive Stress

The bending moment diagram shows positive and negative values. We evaluate stresses at the extreme fibres above and below the neutral axis.

$$\sigma_1 = -\frac{3600 \times 2}{33.33} = -216.2 \text{ (above neutral axis),}$$

$$\sigma_2 = -\frac{2400 \times 4}{33.33} = -288.3 \text{ (below neutral axis).}$$

The largest compressive stress (most negative) is:

$$\sigma_{\max, \text{comp}} = -288.3 \text{ units}$$

Q1. A beam has a rectangular cross-section of width $b = 100 \text{ mm}$ and height $h = 200 \text{ mm}$. The bending moment at a section is $M = 50 \text{ kN}\cdot\text{m}$. Determine the maximum tensile and compressive stresses. [ans: $\sigma_{\max} = \pm 75 \text{ MPa}$]

Q2. For the T-beam shown in the example, if the bending moment is $M = -4000 \text{ N}\cdot\text{m}$, determine the maximum tensile stress. [ans: $\sigma_{\max, \text{tensile}} = 240.2 \text{ MPa}$]

Q3. The bending moment diagram for a simply supported beam has a maximum value $M_{\max} = 20 \text{ kN}\cdot\text{m}$. If the section modulus $Z = 2.5 \times 10^5 \text{ mm}^3$, determine the maximum bending stress that it experiences. [ans: 80 MPa]

4.2 Shear Stresses due to Shear Forces

In the previous sections, we covered the stresses and deformations caused by a bending moment. However, when a load is applied to a beam, it generally produces both a bending moment and a shear force. Whereas the bending moment produces normal stresses, the shear forces produce shear stresses, denoted by τ , acting parallel to the cross-section to maintain equilibrium.

To determine the distribution of shear stress across a cross-section, we use the formula:

$$\tau = \frac{V(x)Q(y)}{It}$$

where:

- $V(x)$ = internal shear force (N)
- I = second moment of area about the neutral axis (m^4)
- t = width of the beam's cross-sectional area at the point of interest (m)
- $Q(y)$ = first moment of area of a portion of the cross-section about the neutral axis (m^3)

We determine Q using:

$$Q = A'\bar{y}$$

where:

- A' = the area above the point where we are determining shear stress (for points above the neutral axis), or below the point (for points below the neutral axis)
- $\bar{y}$ = the distance between the neutral axis and the centroid of region A'

Below is a visualisation of A' and $\bar{y}$ for a rectangular cross-section.

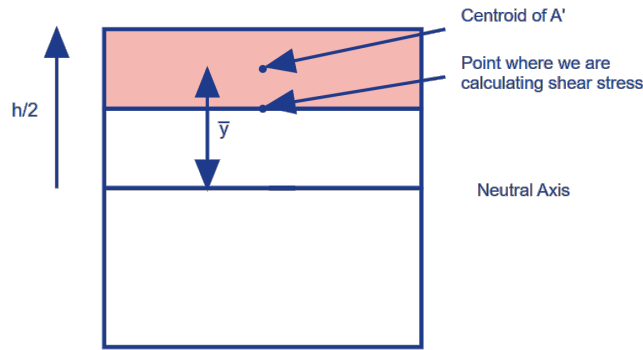


Figure 4.2: Visualisation of A' and $\bar{y}$ for a rectangular cross-section

When taking points at the top or bottom, A' is equal to zero, meaning that the shear stress at the surfaces is at a minimum. At the neutral axis, A' is at a maximum, so the shear stress is also at a maximum here.

In the chapter covering centroids, determining the centroid of a composite shape could be found by splitting it into components and using the formula:

$$\bar{y} = \frac{\sum A_i y_i}{\sum A_i}$$

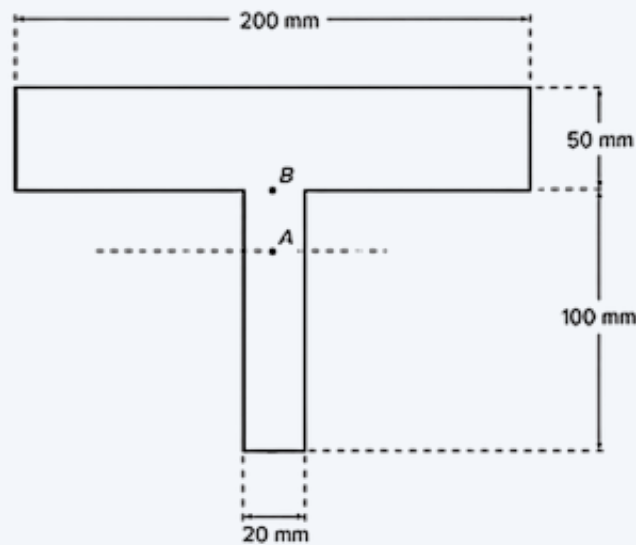
When determining Q for a composite shape, the area can be calculated as the sum of each component's area, $\sum A_i$, simplifying the equation to:

$$Q = \sum A_i \times \bar{y} = \sum A_i y_i$$

This means that for a composite area, you can calculate Q for a certain point by taking the sum of the product of area and distance from the centroid to the neutral axis for each component.

Example 4.2 Finding Shear Stress at Two Points on a T-Beam

Problem: In the T-beam cross-section shown below, point A is at the midpoint of the overall depth and point B is just under the flange. Given that $V(x) = 10\text{N}$, calculate the shear stress at both points.

**Solution: Step 1: Calculating the Centroid**

Split the cross-section into components and calculate their centroids and area to then find the overall centroid

$$y_c = \frac{\sum A_i y_i}{\sum A_i}$$

$$y_c = \frac{(200 \times 50) \times 125 + (100 \times 20) \times 50}{(200 \times 50) + (100 \times 20)} = 112.5$$

The Centroid is located in the top flange. This means A' is the area below the point where we are determining shear stress

Step 2: Calculating the Second Moment of Area

Split the cross-section into two rectangles and calculate each second moment of area with parallel axis correction, then sum them for the total second moment of area.

$$I_1 = \frac{200 \times 50^3}{12} + (200 \times 50)(12.5^2) = 3645833.333 \text{ mm}^4$$

$$I_2 = \frac{20 \times 100^3}{12} + (20 \times 100)(62.5^2) = 9479166.667 \text{ mm}^4$$

$$I_z = I_1 + I_2 = 13.125 \times 10^6 \text{ mm}^4$$

Step 3: Calculating Q

For point B (just under the flange):

$$A'_B = 100 \times 20 = 2000 \text{ mm}^2$$

$$\bar{y}_B = 12.5 \text{ mm}$$

$$Q_B = 2000 \times 12.5 = 25000 \text{ mm}^3 = 2.5 \times 10^4 \text{ mm}^3$$

For point A (located at half the total depth of the shape)

$$A'_A = 75 \times 20 = 1500 \text{ mm}^2$$

$$\bar{y}_A = 75 \text{ mm}$$

$$Q_A = 1500 \times 75 = 112500 \text{ mm}^3$$

Step 3: Convert to Standard Units

$$I_z = 13.125 \times 10^6 \text{ mm}^4 = 1.3125 \times 10^{-5} \text{ m}^4$$

$$Q_B = 25000 \text{ mm}^3 = 2.5 \times 10^{-5} \text{ m}^3 \quad Q_A = 112500 \text{ mm}^3 = 1.125 \times 10^{-4} \text{ m}^3$$

$$t = 20 \text{ mm} = 0.02 \text{ metres}$$

Step 3: Calculating the Shear Stress

The width t at both points is 20 mm (the web thickness).

$$\tau_B = \frac{V(x)Q_B}{It} = \frac{10 \times 2.5 \times 10^{-5}}{1.3125 \times 10^{-5} \times 0.02} = 0.0083 \text{ N/mm}^2 = 952.4 \text{ Pa}$$

$$\tau_A = \frac{V(x)Q_A}{It} = \frac{10 \times 1.125 \times 10^{-4}}{1.3125 \times 10^{-5} \times 0.02} = 0.0084375 \text{ N/mm}^2 = 4285.7 \text{ Pa}$$

$$\boxed{\tau_B = 952.4 \text{ Pa}}, \quad \boxed{\tau_A = 4285.7 \text{ Pa}}$$

Q1. A rectangular beam has width $b = 100 \text{ mm}$ and height $h = 200 \text{ mm}$. If the shear force is $V = 20 \text{ kN}$, calculate the maximum shear stress. [ans: 1.5 MPa]

Q2. For the T-beam in the example, calculate the shear stress at a point 30 mm below the top of the flange. [ans: 514.3 Pa]

Q3. A beam has an I-section with dimensions: flanges $150 \times 20 \text{ mm}$, web $20 \times 300 \text{ mm}$. If $V = 15 \text{ kN}$, calculate the maximum shear stress. [ans: 2.66 MPa]

Q4. Derive the expression for the maximum shear stress in a rectangular cross-section in terms of the average shear stress. [ans: $\tau_{\max} = 1.5\tau_{\text{avg}}$]

Shear Stress Distribution in Common Cross-Sections

For different cross-sectional shapes, the shear stress distribution varies. Understanding this distribution is crucial for determining the maximum shear stress and ensuring the beam's structural integrity.

Rectangular Cross-Section:

For a rectangular cross-section of width b and height h , the shear stress distribution

is parabolic, with the maximum occurring at the neutral axis:

$$\tau_{\max} = \frac{3V}{2bh} = \frac{3}{2} \tau_{\text{avg}}$$

Circular Cross-Section:

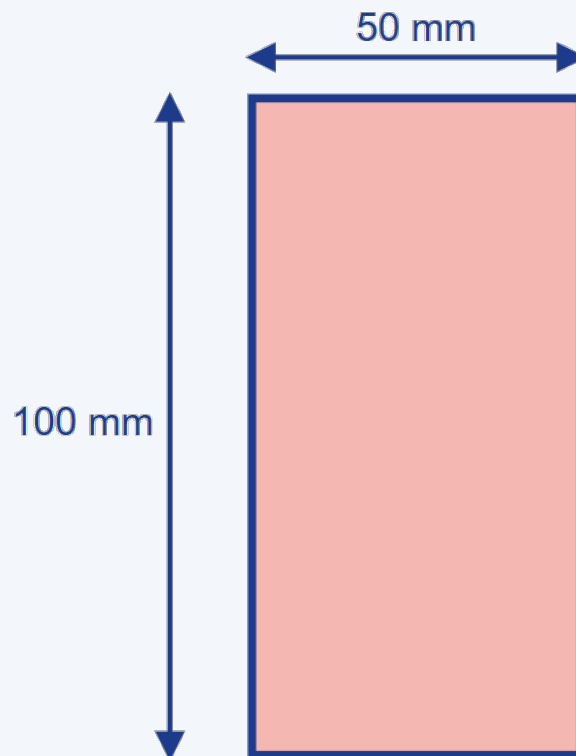
For a circular cross-section of radius R , the maximum shear stress occurs at the neutral axis and is given by:

$$\tau_{\max} = \frac{4V}{3\pi R^2} = \frac{4}{3} \tau_{\text{avg}}$$

In general, the maximum shear stress in a beam cross-section occurs at the neutral axis, where Q is maximum, except for cross-sections where the width t at the neutral axis is unusually large (e.g., some I-beams where the maximum may occur in the web just below the flange).

Example 4.3 Shear Stress Distribution in a Rectangular Beam

Problem: A rectangular beam has width $b = 50 \text{ mm}$ and height $h = 100 \text{ mm}$. The beam is subjected to a shear force $V = 5 \text{ kN}$. Determine the shear stress at the neutral axis and at a point 25 mm above the neutral axis.



Solution:

Step 1: Calculate the second moment of area

$$I = \frac{bh^3}{12} = \frac{50 \times 100^3}{12} = 4.1667 \times 10^6 \text{ mm}^4$$

Step 2: Calculate Q at the neutral axis

$$Q_{\max} = A'\bar{y} = (50 \times 50) \times 25 = 62500 \text{ mm}^3$$

Step 3: Calculate τ at the neutral axis

$$\tau_{\max} = \frac{VQ}{It} = \frac{5000 \times 62500}{4.1667 \times 10^6 \times 50} = 1.5 \text{ N/mm}^2 = 1.5 \text{ MPa}$$

Step 4: Calculate Q at a point 25 mm above the neutral axis

$$Q = (25 \times 50) \times 37.5 = 46875 \text{ mm}^3$$

Step 5: Calculate τ at $y = 25 \text{ mm}$

$$\tau = \frac{5000 \times 46875}{4.1667 \times 10^6 \times 50} = 1.125 \text{ N/mm}^2 = 1.125 \text{ MPa}$$

$$\tau_{\max} = 1.5 \text{ MPa},$$

$$\tau_{y=25} = 1.125 \text{ MPa}$$

Q1. A circular beam has a diameter of $d = 100 \text{ mm}$. If the shear force is $V = 10 \text{ kN}$, calculate the maximum shear stress. [ans: 1.70 MPa]

Q2. For the rectangular beam in Example 4.3, determine the shear stress at a point 40 mm above the neutral axis. [ans: 0.54 MPa]

4.3 Beam Deflection due to Bending Moment

Beam deflection, denoted by v , describes the displacement of the beam from its original position due to the bending moment deforming it. The magnitude of deflection depends on the setup of the beam, the material's resistance to deformation, and the distribution of the applied load. Beam deflection allows us to assess whether a material has enough stiffness for its intended application. The formula relating the bending moment to beam deflection is:

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

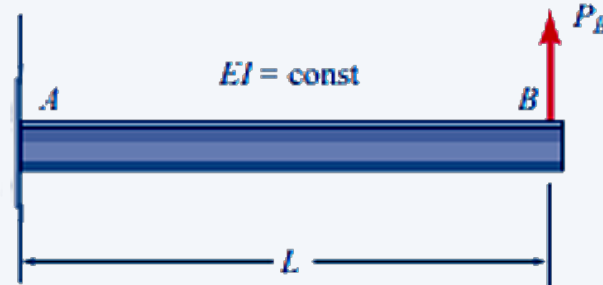
where:

- M = bending moment (N·m)
- E = Young's modulus (Pa)
- I = second moment of area (m^4)
- x = distance along the beam (m)

Each deflection equation must be derived using double integration. The first step is determining the bending moment as a product of force and distance, then substituting it back into the equation. The equation is integrated twice, and the constants are found using boundary conditions that depend on the support type. For a fixed support, the deflection v and the slope dv/dx are both zero. For pin and roller supports, $v = 0$ but dv/dx is generally not zero.

Example 4.4 Deflection of a Cantilever Beam with a Point Load at the Free End

Problem: A cantilever beam of length L is fixed at A and carries a point load P at its free end B , as shown below. Derive the deflection equation and find the maximum deflection.

**Solution:**

Taking x as the distance from the fixed end A , the bending moment at a section is:

$$M(x) = -P(L - x)$$

With the sign convention that positive moment causes sagging (compression top), we have:

$$EI \frac{d^2v}{dx^2} = -P(L - x)$$

Integrate once:

$$EI \frac{dv}{dx} = -P \left(Lx - \frac{x^2}{2} \right) + C_1$$

Integrate again:

$$EIv = -P \left(\frac{Lx^2}{2} - \frac{x^3}{6} \right) + C_1x + C_2$$

Boundary conditions: at the fixed end $x = 0$, $v = 0$ and $dv/dx = 0$.

$$EI(0) = -P(L \cdot 0 - 0) + C_1 \Rightarrow C_1 = 0,$$

$$EI(0) = -P \left(\frac{L \cdot 0^2}{2} - \frac{0^3}{6} \right) + C_1 \cdot 0 + C_2 \Rightarrow C_2 = 0.$$

Thus,

$$v(x) = -\frac{P}{EI} \left(\frac{Lx^2}{2} - \frac{x^3}{6} \right)$$

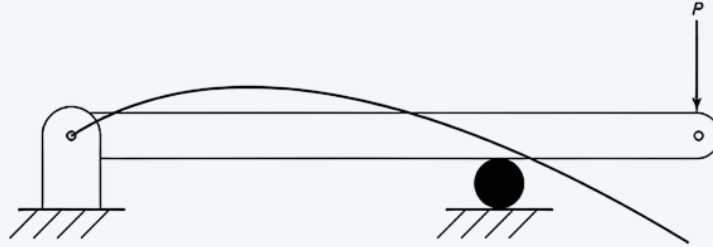
The maximum deflection occurs at the free end $x = L$:

$$v_{\max} = v(L) = -\frac{P}{EI} \left(\frac{L^3}{2} - \frac{L^3}{6} \right) = -\frac{PL^3}{3EI}$$

$$v_{\max} = -\frac{PL^3}{3EI}$$

Example 4.5 Deflection of a Simply Supported Beam with an Overhang

Problem: Consider the beam shown below. Let x be the distance from the pin support at A , with the roller support at B located at $x = 2$ m, and the total beam length is 3 m. A point load $P = 4$ N acts at the free end C ($x = 3$ m). Determine the equations for deflection in each region.



Solution:

Determine support reactions:

Taking moments about A :

$$-R_B \times 2 + 4 \times 3 = 0 \Rightarrow R_B = 6 \text{ N (upward)}$$

Summing vertical forces:

$$R_A + R_B - 4 = 0 \Rightarrow R_A = -2 \text{ N (downward)}$$

Region 1: $0 \leq x \leq 2$

For a section in this region, the bending moment is:

$$M(x) = R_A x = -2x$$

Thus,

$$EI \frac{d^2 v_1}{dx^2} = -2x$$

Integrate:

$$EI \frac{dv_1}{dx} = -x^2 + C_1, \quad EI v_1 = -\frac{x^3}{3} + C_1 x + C_2$$

Boundary conditions: at $x = 0$, $v_1 = 0 \Rightarrow C_2 = 0$; at $x = 2$, $v_1 = 0 \Rightarrow -\frac{8}{3} + 2C_1 = 0 \Rightarrow C_1 = \frac{4}{3}$.

$$v_1(x) = \frac{1}{EI} \left(-\frac{x^3}{3} + \frac{4x}{3} \right) = \frac{4x - x^3}{3EI}$$

Region 2: $2 \leq x \leq 3$

For a section in this region, the bending moment is (considering forces to the left):

$$M(x) = R_A x + R_B(x - 2) = -2x + 6(x - 2) = 4x - 12$$

Thus,

$$EI \frac{d^2 v_2}{dx^2} = 4x - 12$$

Integrate:

$$EI \frac{dv_2}{dx} = 2x^2 - 12x + C_3, \quad EI v_2 = \frac{2}{3}x^3 - 6x^2 + C_3x + C_4$$

Continuity conditions at $x = 2$:

$$v_1(2) = v_2(2) = 0 \Rightarrow \frac{2}{3}(8) - 6(4) + C_3(2) + C_4 = 0,$$

$$\frac{dv_1}{dx}(2) = \frac{dv_2}{dx}(2) \Rightarrow -4 + \frac{4}{3} = 8 - 24 + C_3 \Rightarrow C_3 = \frac{40}{3}$$

Then C_4 :

$$\frac{16}{3} - 24 + \frac{80}{3} + C_4 = 0 \Rightarrow C_4 = -8$$

Hence,

$$v_2(x) = \frac{1}{EI} \left(\frac{2}{3}x^3 - 6x^2 + \frac{40}{3}x - 8 \right)$$

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Standard Formulas for Maximum Deflection

For common loading and support conditions, the maximum deflection can be found using the following standard results.

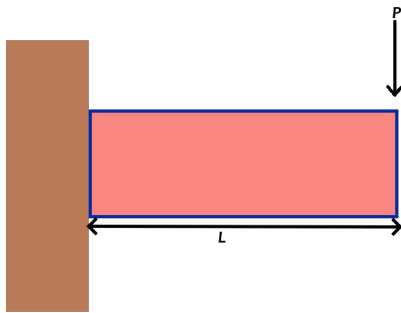


Figure 4.3: Cantilever with point load at free end: $v_{\max} = -\frac{PL^3}{3EI}$

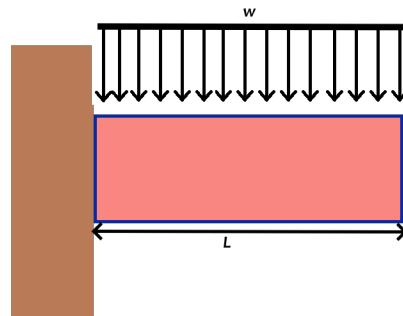


Figure 4.4: Cantilever with uniformly distributed load: $v_{\max} = -\frac{wL^4}{8EI}$

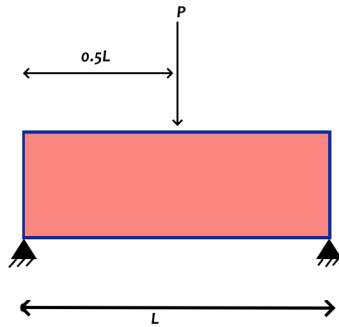


Figure 4.5: Simply supported with central point load: $v_{\max} = -\frac{PL^3}{48EI}$

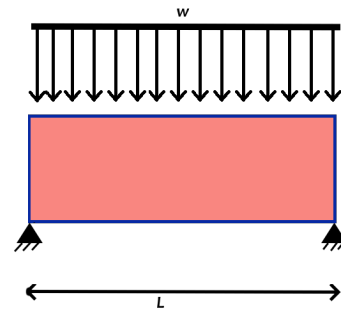


Figure 4.6: Simply supported with uniformly distributed load: $v_{\max} = -\frac{5wL^4}{384EI}$

Q1. A cantilever beam of length $L = 2$ m has a rectangular cross-section $b = 50$ mm, $h = 100$ mm and is made of steel with $E = 200$ GPa. A point load of $P = 10$ kN acts at the free end. Determine the maximum deflection. [ans: 32.0 mm]

Q2. A simply supported beam of span $L = 4$ m carries a uniformly distributed load $w = 5$ kN/m. If $EI = 2 \times 10^7$ N·m², calculate the maximum deflection. [ans: 0.83 mm]

Q3. Derive the deflection equation for a simply supported beam with a central point load P . Show that the maximum deflection is $-\frac{PL^3}{48EI}$. [Derivation required]

4.4 Chapter Review

Mixed Problems

Problem 4.4.1 A beam has a rectangular cross-section of width $b = 150 \text{ mm}$ and height $h = 250 \text{ mm}$. The bending moment is $M = 30 \text{ kN}\cdot\text{m}$ and the shear force is $V = 20 \text{ kN}$. Determine:

- (a) The maximum normal stress
- (b) The maximum shear stress
- (c) The normal and shear stresses at a point 50 mm above the neutral axis

Problem 4.4.2 For the T-beam shown in Example 4.2, determine the shear stress at a point 10 mm below the top of the flange.

Problem 4.4.3 An I-beam has dimensions: top flange $200 \times 25 \text{ mm}$, bottom flange $200 \times 25 \text{ mm}$, web $15 \times 350 \text{ mm}$. If $V = 40 \text{ kN}$, calculate:

- (a) The second moment of area
- (b) The maximum shear stress
- (c) The shear stress in the web just below the flange

Problem 4.4.4 Derive the deflection equation for a cantilever beam with a uniformly distributed load over its entire length. Find the maximum deflection.

Problem 4.4.5 A simply supported beam of length L carries a point load P at a distance a from the left support. Derive the deflection equations for each region and find the maximum deflection.

Problem 4.4.6 A beam with a T-shaped cross-section (same as in Example 4.1) is subjected to a bending moment of $M = 30 \text{ kN}\cdot\text{m}$ and a shear force of $V = 15 \text{ kN}$. Calculate the maximum normal stress, maximum shear stress, and the shear stress at the junction of the flange and web.

Problem 4.4.7 Using the double integration method, determine the slope and deflection at the free end of a cantilever beam carrying a moment M_0 at the free end.

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Beam Bending Summary

Normal Stress due to Bending:

- Formula: $\sigma = \frac{My}{I}$
- Maximum stress occurs at the extreme fibres ($y = \pm c$).
- Requires determination of centroid and second moment of area.

Shear Stresses due to Shear Forces:

- Formula: $\tau = \frac{VQ}{It}$
- $Q = A'\bar{y}$ (first moment of area)
- Maximum shear stress occurs at the neutral axis.
- Rectangular: $\tau_{\max} = \frac{3V}{2bh}$
- Circular: $\tau_{\max} = \frac{4V}{3\pi R^2}$

Beam Deflection:

- Derived from $\frac{d^2v}{dx^2} = \frac{M}{EI}$.
- Double integration yields $v(x)$ and constants from boundary conditions.
- Common boundary conditions: fixed end $v = 0, v' = 0$; pin/roller $v = 0$.
- Standard formulas for common load cases.

Key Equations:

- Normal stress: $\sigma = \frac{My}{I}$
- Shear stress: $\tau = \frac{VQ}{It}$
- Deflection: $\frac{d^2v}{dx^2} = \frac{M}{EI}$
- Cantilever, point load: $v_{\max} = -\frac{PL^3}{3EI}$
- Simply supported, central load: $v_{\max} = -\frac{PL^3}{48EI}$

Chapter Checklist

- ☐ I can calculate the centroid of a composite cross-section.
- ☐ I can compute the second moment of area using the parallel axis theorem.
- ☐ I can determine the bending stress at any point in a beam cross-section.
- ☐ I can calculate the first moment of area Q for a given cross-section.
- ☐ I can determine the shear stress at any point in a beam cross-section.
- ☐ I can identify the location of maximum shear stress.

- ☐ I can derive the deflection equation using double integration.
- ☐ I can apply boundary conditions to find constants of integration.
- ☐ I can use standard deflection formulas for common loading cases.
- ☐ I can set up and solve deflection problems with multiple regions.
- ☐ I can calculate both normal and shear stresses at any point.

